# SPARSE RECOVERY: FUNDAMENTAL LIMITS, MEASUREMENT CONSTRUCTIONS AND GRAPH CONSTRAINTS

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SPARSE RECOVERY: FUNDAMENTAL LIMITS, MEASUREMENT CONSTRUCTIONS AND GRAPH CONSTRAINTS

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Sparse recovery explores the sparsity structure inside data and aims to find a low-dimensional representation for a high-dimensional sparse object. Since some form of signal sparsity naturally exists in many applications, sparse recovery can benefit areas like imaging, communication, network monitoring, etc. There has been an exploration of research on the topic *compressed sensing*, which indicates that an incomplete set of linear projections can represent high-dimensional sparse signals, and the unknown sparse signal can be efficiently recovered by $\ell_1$-minimization.

$\ell_1$-minimization can be viewed as a convex relaxation of a NP-hard $\ell_0$-minimization problem, and its sparse recovery performance has been characterized and extensively analyzed in the literature of compressed sensing. $\ell_p$-minimization ($p \in [0, 1)$) returns a vector with the least $\ell_p$ quasinorm among all the vectors that can produce the same linear measurements. Though computationally more expensive to solve, $\ell_p$-minimization is generally believed to have a better sparse recovery performance than $\ell_1$-minimization. In Chapter 2, we investigate the sparse recovery ability of $\ell_p$-minimization. When the measurement matrices are Gaussian, we provide sharp thresholds of the sparsity ratio (percentage of nonzero entries of a vector) that differentiates the success and failure of sparse recovery. We consider its strong recovery performance which requires to recover all the sparse vectors up to certain sparsity; and we also

for the first time analyze its weak recovery performance which aims to recover all the sparse vectors on one support with a fixed sign pattern. Surprisingly, our results indicate that although the strong recovery performance improves as $p$ decreases, $\ell_1$-minimization has the best weak recovery performance for all $p$ between zero and one.

The efficient administration of communication networks relies on accurate estimates of network characteristics such as transmission rates and link queueing delays. Since measuring each component in the network directly can be operationally costly, or even infeasible, one needs to infer system internal characteristics from indirect end-to-end (aggregate) measurements. This topic is known as *network tomography*. It has a natural connection to sparse recovery, since many network parameters are indeed sparse, e.g., link delays.

The marriage of network tomography and sparse recovery offers new directions to explore. In network applications, each measurement should satisfy the network topological constraints such as forming a feasible path or a cycle in a given network topology. Most measurement constructions in sparse recovery, however, assume that any subset of the values can be aggregated together in a measurement. In Chapter 3, we consider constructions of sparse recovery measurements with additional graph topological constraints. Explicit measurement constructions for various graphs are provided, and the number of the constructed measurements is less than the existing estimate of the number of measurements required to recover sparse vectors over graphs. We also propose a measurement construction algorithm and characterize the dependence of the number of measurements required for sparse recovery on the graph structure.

Some network parameters such as link delays are nonnegative. A nonnegative sparse signal can be the only nonnegative solution to an underdetermined linear system. In Chapter 4, we discuss this uniqueness property for binary measurement matrices and prove that a sparse vector is a unique nonnegative solution even if its support size is proportional to the dimension.

## BIOGRAPHICAL SKETCH

Meng Wang was born in Changzhou, Jiangsu, China. Meng received the Bachelor's Degree and the Master's Degree (both with honors) in Electrical Engineering from Tsinghua University, Beijing, China, in 2005 and 2007.

Since August 2007, Meng has been a PhD student in the School of Electrical and Computer Engineering at Cornell University. Her research interests include communication networks, signal processing, and nonconvex optimization and its applications.

Meng is a recipient of Jacobs Fellowship of Cornell University in 2008 and 2010.

To my family.

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CHAPTER 1

# INTRODUCTION

## 1.1 Sparse recovery

### 1.1.1 Motivation

The Nyquist-Shannon sampling theorem is a fundamental result in the field of information theory and signal processing. It indicates that in order to correctly recover continuous-time bandwidth-limited signals from an infinite sequence of uniformly spaced discrete samples, the sampling rate should be no less than the *Nyquist rate*, which is twice the highest frequency in the signal to recover. In many applications, it can be very costly, or even physically impossible [129], to achieve the Nyquist rate. Even if the desired sampling rate is achieved, that would be a large amount of data to process.

Compressibility or sparsity plays an important role in the efficient data acquisition process. A signal of length $n$ is *compressible* or *sparse* if it can be represented by $k$ ($k \leq n$) coefficients. For example, for a typical image, only a small fraction of its wavelet coefficients are significant, while other wavelet coefficients are relatively small and can be thrown away in the reconstruction process without introducing much perceptual loss. Such compressibility or signal sparsity is exploited in modern transform coding schemes including JPEG, JPEG2000, and MPEG. A typical process is that one fully samples the signal, computes the complete set of transform coefficients, keeps the largest coefficients and discards all the others. This is extremely wasteful since after acquiring massive amounts of data, we discard most of them in the subsequent compression stage.

*Compressed sensing*, also referred to as *compressive sensing* or *compressive sampling* has emerged as a new framework for data acquisition. Instead of first fully sampling and then throwing away most data in compression, compressed sensing explores the compressible structure of signals and directly samples the signals in a compressed form, for example, at a lower sampling rate than Nyquist rate. It only takes a small number of *nonadaptive* linear measurements, yet it promises to recover high-dimensional signals accurately. For an $n$-dimensional signal, in general we need at least $n$ linear measurements to reconstruct it. However, if a signal is sparse in some basis, compressed sensing theory indicates that one can hope to recover it from $m$ ($m \ll n$) linear measurements. As we do not know the locations of the significant coefficients of a signal, it is a highly nontrivial task to design a small set of linear measurements such that the sparse signal can be correctly recovered by certain reconstruction method.

### 1.1.2 Mathematical Formulation

Let me first introduce the mathematical model that will be discussed here. The unknowns signal of interest is represented by a vector $\mathbf{x}$ in $\mathcal{R}^n$. Its support $T$ characterizes the locations of the non-zero entries, and is defined as

$$T := \{i \in \{1, ..., n\} : x_i \neq 0\}.$$

We say an $n$-dimensional vector $\mathbf{x}$ is $k$-sparse if it has $k$ non-zero entries, i.e., the cardinality of set $T$ satisfies $|T| = k$. The sparsity of $\mathbf{x}$ is also represented by its $\ell_0$

norm:

$$\|\mathbf{x}\|_0 := |\{i : x_i \neq 0\}| = |T|.$$

In the literature of compressed sensing, we take $m$ ($m \ll n$) nonadaptive linear measurements of $\mathbf{x}$, and $A^{m\times n}(m < n)$ is the real-valued measurement matrix. The $i$th measurement is the inner product between $\mathbf{x}$ and the $i$th row vector of $A$. Let $\mathbf{y}$ in $\mathcal{R}^m$ represent the vector of measurement, then

$$\mathbf{y} = A\mathbf{x}. \tag{1.1}$$

The goal is to recover $\mathbf{x}$ given $\mathbf{y}$ and $A$. Clearly, when $m < n$, $A\mathbf{x} = \mathbf{y}$ is an underdetermined linear system and admits an infinite number of solutions. However, if $\mathbf{x}$ is sparse in some known basis, i.e., its coefficients in that basis are mostly zero, then one can indeed correctly recover $\mathbf{x}$ from $\mathbf{y}$. Throughout the dissertation we assume $\mathbf{x}$ itself to be sparse without loss of generality. If $\mathbf{x}$ is sparse in some other basis, i.e., vector $\mathbf{z} = \Phi\mathbf{x}$ is sparse, where $\Phi^{n\times n}$ is an invertible matrix, then

$$\mathbf{y} = A\mathbf{x} = A\Phi^{-1}\mathbf{z} = \Psi\mathbf{z}, \tag{1.2}$$

where $\Psi := A\Phi^{-1}$. Thus, (1.2) shares the same form as (1.1).

The problem formulation of compressed sensing is closely related to, but different from that of *(combinatorial) group testing* [57, 58], while they both explore the sparsity structure of the signals to achieve reduction in the number of measurements needed. Group testing started during the Second World War, and its motivation is to do large scale blood testing economically [57]. It is also important in applications including industrial testing [117], data compression [78], DNA library screening [101], multiple access control protocols [83, 134], and data streams [39]. There are two main differences between compressed sensing and group testing models. First, in compressed sensing, the signal $\mathbf{x}$, the measurement $\mathbf{y}$ and the measurement matrix $A$ are real-valued. But in group testing, they are all logical, taking values either '0' or '1', i.e., $\mathbf{x} \in \{0,1\}^n$, $\mathbf{y} \in \{0,1\}^m$, and $A \in \{0,1\}^{m\times n}$. Second, in compressed sensing, each measurement is an inner product between the corresponding row of $A$ and $\mathbf{x}$, while in group testing, the operations are replaced with logical "AND" and "OR". Though the models are different, both compressed sensing and group testing aim to design a small number of measurements (either real or logical) such that all the vectors (either real or logical) up to certain sparsity can be correctly recovered by certain reconstruction scheme. Here we consider compressed sensing setup if not otherwise specified and will occasionally discuss group testing for comparison.

One key question in compressed sensing is how we can correctly recover $n$-dimensional $k$-sparse vector $\mathbf{x}$ from $m$-dimensional measurement $\mathbf{y}$. One natural estimate of $\mathbf{x}$ is the vector with the least $\ell_0$-norm that can produce the measurement $\mathbf{y}$. Mathematically, to recover $\mathbf{x}$, we solve the following $\ell_0$-minimization problem:

$$\min_{\mathbf{x}\in\mathcal{R}^n} \|\mathbf{x}\|_0 \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{y}. \tag{1.3}$$

In fact, if every $2k$ columns of $A$ are linearly independent, a $k$-sparse vector $\mathbf{x}$ is indeed the solution to (1.3). To see this, let $\mathbf{z}$ denote the solution to (1.3), then $\|\mathbf{z}\|_0 \le \|\mathbf{x}\|_0 = k$. Since $A\mathbf{z} = A\mathbf{x} = \mathbf{y}$, we have $A(\mathbf{z}-\mathbf{x}) = \mathbf{0}$, i.e., $\mathbf{z}-\mathbf{x}$ is in the null space of $A$. Since $\mathbf{z}-\mathbf{x}$ has at most $2k$ non-zero entries, and every $2k$ columns of $A$ are linearly independent, $\mathbf{z} = \mathbf{x}$ must hold.

(1.3) is a natural method to recover sparse signals, however, it is combinatorial and computationally intractable to solve in general. In practice, one commonly used approach is to solve a closely related $\ell_1$-minimization problem where we replace the $\ell_0$-norm of (1.3) with the $\ell_1$-norm. Mathematically, we solve

$$\min_{\mathbf{x}\in\mathcal{R}^n} ||\mathbf{x}||_1 \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{y}, \tag{1.4}$$

where $||\mathbf{x}||_1 := \sum_i |x_i|$. (1.4) is a convex problem and can be recast as a linear program, thus can be solved in time $O(n^3)$[1]. (1.4) is referred to as $\ell_1$-minimization or Basis Pursuit, which has been extensively studied in the literature of compressed sensing. The breakthrough results by Candès and Tao [20, 21] and Donoho [47] indicate that even though $\ell_1$-minimization is a convex relaxation of the $\ell_0$-minimization, it is guaranteed to accurately recover the sparse vector from a small number of linear measurements.

### 1.1.3 Recovery Methods

As reviewed in [22], Logan [90] already observed the sparsity-promoting feature of $\ell_1$-norm in 1960s, and the analytical results were later presented by Donoho and Stark [50], and Donoho and Logan [52] in late 1980s. On the application side, in 1970s, $\ell_1$-minimization was proposed in reflection seismology to deconvolve seismic traces so as to determine marine surface structures [122] and was later refined to handle observation noise [112]. In 1990s, in the context of computational harmonic analysis, Basis Pursuit [31, 113] was proposed to decompose a signal into a sparse superposition of elements from a overcomplete dictionary. Tibshirani proposed the least absolute shrinkage and selection operator (LASSO) [123], an $\ell_1$-regularized $\ell_2$-minimization problem, for sparse model fitting in statistics. At this point, most results were mainly empirical.

[1] $g(n) \in O(h(n))$ if as $n$ goes to infinity, $g(n) \leq ch(n)$ eventually holds for some constant $c > 0$.

It then came an exploration of theoretical results in 2000s. $\ell_1$-minimization was first proved to be able to correctly recover sparse signals from an incomplete set of linear measurements [20, 21, 19, 47, 53]. Clearly the recovery performance depends on the choice of the measurement matrix $A$. The Restricted Isometry Property (RIP) conditions were introduced in [20, 21], and Candès and Tao showed that if matrix $A$ satisfies RIP conditions, then $\ell_1$-minimization can correctly recover all sparse signals. Random matrices with independent zero-mean Gaussian entries or independent Bernoulli entries are proven to satisfy the RIP conditions with overwhelming probability [7, 20]. RIP conditions are sufficient conditions for the successful sparse recovery via $\ell_1$-minimization. Using tools in high-dimensional geometry, Donoho and Tanner [53, 54] proved that a necessary and sufficient condition for $\ell_1$-minimization to recover $k$-sparse signals is that $A$ constitutes a *k-neighborly* polytope. If matrix $A^{m \times n}$ has i.i.d. Gaussian random entries, tight bounds of the relation between $k$ and $m$ (given $n$) were developed so as to characterize the neighborliness property as well as the success of sparse recovery via $\ell_1$-minimization [53]. For Gaussian random matrices, with overwhelming probability, $\ell_1$-minimization can recover all $n$-dimensional $k$-sparse vectors provided that the number of measurements satisfies $m = O(k \log(n/k))$ [20, 110].

The $\ell_1$-minimization approach is based on linear programming, which has running time $O(n^3)$ and may not be sufficiently fast in large scale problems. Researchers also developed many greedy algorithms which in general run faster than $\ell_1$-minimization. Matching Pursuit [95] and a refined version Orthogonal Matching Pursuit (OMP)[105] were proposed in 1990s. Various researchers studied the recovery performance of OMP analytically, e.g., [44, 86, 124, 125]. Tropp and Gilbert [125] proved that with high probability, OMP can recover a

$n$-dimensional $k$-sparse signal from $m = O(k \log n)$ measurements. The number of measurements required by OMP is slightly larger than that needed by $\ell_1$-minimization. Moreover, unlike the uniform guarantee of recovering *all* sparse signals by $\ell_1$-minimization, OMP only guarantees to recover a *fixed* sparse signal [107, 125]. Variations of OMP include Simultaneous Orthogonal Matching Pursuit(S-OMP) [126], Stagewise Orthogonal Matching Pursuit (StOPM), Regularized Orthogonal Matching Pursuit [100], as well as Compressive Sampling Matching Pursuit (CoSaMp)[99], which has a fast running time $O(n \log^2 n)$ for sparse signals. Iterative Thresholding [13, 41, 67] and Iterative Least Squares [29, 42] are also examples of greedy algorithms.

Combinatorial algorithms take advantage of the special structures of the measurement matrix $A$ to achieve rapid reconstruction. They are faster than greedy algorithms, but usually require a large number of highly structured measurements. Fourier Sampling [70], Chaining Pursuit [71], Expander Matching Pursuit [80], as well as [3, 40, 79, 91, 138] are examples of a plethora of combinatorial algorithms.

### 1.1.4 Applications

Sparse recovery has wide applications in compressive imaging, medical imaging, analog-to-information (A/D) conversion, biology, networks, to name a few.

As mentioned earlier, many images are sparse or relatively sparse in some basis, e.g., smooth images are sparse in the Fourier basis. Instead of first sensing every pixel of an image and then discarding most data away after compression as in today's digital cameras, compressive imaging [128] directly acquires the linear projections of an image and then applies the compressed sensing reconstruction algorithm. The number of projections is much less than the number of pixels of an image, and therefore the computation for data acquisition is significantly reduced. A prototype "single-pixel" camera was built under this framework [59]. Compresses sensing is also very useful in medical imaging such as magnetic resonance imaging(MRI), e.g., [93, 94]. It greatly reduces scan time, which in turn reduces the patients' exposure to stimulating signals, as well as the costs of MRI. The theory of sparse recovery enables direct A/D conversion of compressible signals at sub-Nyquist rates, and many efforts have been contributed to the implementation of high-rate A/D converters, e.g., [87, 97]. In biological applications, group testing [57] was proposed in World War II to do blood testing in soldiers. As the percentage of infected soldiers was very small, a huge number of expensive tests could be saved by grouping blood samples and testing a combined sample each time. Recently, compressed sensing idea was incorporated into the study of gene expression level to create the so-called "compressed microarrays" [104, 114]. DNA microarrays are collections of microscopic DNA spots that can detect and measure the expression levels of large numbers of genes simultaneously. With compressed microarrays, each spot can measure a linear combination of several gene expression levels, and the number of spots is potentially much smaller than the number of genes being tested.

The application of sparse recovery in networks caught researchers' attention recently. Many network characteristics are sparse or approximately sparse, e.g., link transmission delays in the Internet, link failures in all-optical networks, connectivity patterns in the wireless sensor networks, and structures of social networks can all be represented by sparse signals. Thus, failure localization in all-optical networks can be formulated as a group testing problem with network topological constraints [4, 33, 75, 121, 136], and compressed sensing is connected to network monitoring in the communication networks [36, 66, 76, 142], neighbor discovery in ad-hoc networks [143], and clique identification in social networks [81].

The applications of sparse recovery are not limited to the above examples. For interested readers, please refer to [38] for details about many other applications.

## 1.2 Some Important Questions

Sparse recovery has received tremendous attention after the breakthrough [20, 21, 47, 53] in 2000s. There are many interesting theoretical and practical problems to address in this field.

### 1.2.1 Performance Analysis of Sparse Recovery Methods

Many recovery algorithms have been developed for sparse recovery, and it is necessary to analyze and characterize their recovery performance. For a particular recovery algorithm together with a measurement construction method, one important quantity for performance analysis is the recovery threshold, which is the largest number of non-zero elements that the unknown signal of interest can have such that it is still guaranteed to be correctly recovered. When the signals are approximately sparse or the measurements are corrupted with noise, the stability and robustness of these algorithms also need to be analyzed.

After $\ell_1$-minimization was first proved [19, 20, 21, 47, 53] to be able to recover sparse signals, much work has been devoted to the theoretical analysis of $\ell_1$-minimization and has yielded tremendous results. [8, 37, 48, 51, 56, 82, 89, 119, 141, 118, 120, 144] are a few examples of excellent references. There are performance analysis about some other recovery methods, such as [125] for OMP and [99] for CoSaMp. But in general, analytical results about the recovery guarantee and stability of other recovery methods are still limited.

### 1.2.2 Construction of Measurement Matrices

The sparse recovery performance also depends on the chosen measurement matrix. Theoretical analysis mostly focus on random matrices such as random Gaussian matrices, random Bernoulli matrices, and random Fourier matrices. With overwhelming probability, $\ell_1$-minimization can recover $n$-dimensional $k$-sparse signals from $m = O(k \log(n/k))$ random measurements [20, 110]. The random matrices are easier to analyze, but have practical limitations. Random construction is not guaranteed to produce a "good" measurement matrix for sparse recovery. Although a randomly generated matrix satisfies the requirement for sparse recovery with high probability, there is no fast algorithm to check whether or not a given matrix is indeed a "good" measurement matrix. Therefore, it is necessary to develop deterministic constructions of good measurement matrices.

Some measurement constructions methods such as [10, 8, 80, 85, 138] are based on bipartite expander graphs [25, 115], which can be obtained either randomly or explicitly. See, for example, [18] for random constructions and [25] for explicit constructions of expander graphs. Other explicit construction methods include but not limited to [2, 3, 5, 40, 46, 84]. The number of measurements required by explicit constructions to recover $n$-dimensional $k$-sparse signals is much greater than $O(k \log(n/k))$ in general. It is still an open problem to explicitly construct measurements such that the number of observations required by sparse recovery is comparable to that required by random constructions.

Moreover, current construction methods critically rely on the assumption that a measurement can be a linear combination of any subset of the entries of the unknown signal of interest. In applications such as network monitoring, however, a measurement should satisfy additional constraints [142, 130], e.g., forming a feasible path in the network. Construction of sparse recovery measurements with additional practical constraints is interesting to explore.

### 1.2.3 Fast Recovery Algorithms with Performance Guarantee

One major recovery technique in sparse recovery is $\ell_1$-minimization, also known as basis pursuit. $\ell_1$-minimization has theoretical performance guarantee, but is not optimally fast in application ($O(n^3)$ time complexity to be precise). Greedy algorithms (e.g. OMP [95, 105, 125] and CoSaMp[99]) and combinatorial algorithms (e.g. Chaining Pursuit [71]) can run faster than $\ell_1$-minimization, but they may require more measurements than $\ell_1$-minimization, and/or do not have recovery guarantee and stability results. Therefore, it would be interesting to design sparse recovery algorithms as well as the corresponding measurement matrices such that (1) they have provable recovery guarantee; (2) the number of required measurements is comparable to that required by $\ell_1$-minimization; (3)

they have low computational complexity.

### 1.2.4 Existence of Low-dimensional Representation

Sparse recovery indicates that $m$ ($m \leq n$) linear projections can characterize a vector in $\mathcal{R}^n$ as long as the vector is sparse. In fact, sparsity is only one type of geometry that allows a low-dimensional representation of a high-dimensional object. For example, recent research suggested that the low rank property in the matrix space also guarantees low-dimensional representations of high-order matrices. See [23, 64, 108, 109] as examples of many interesting results. Then does there exist other geometry that allows a low-dimensional representation of a high-dimensional object? If so, how could we find such a low-dimensional representation efficiently?

## 1.3 Main contributions

This dissertation considers the fundamental limits of sparse recovery methods other than $\ell_1$-minimization. Motivated by network applications, it also addresses the problem of explicit constructions of sparse recovery measurements in the presence of additional practical constraints. It also studies the special property of sparse nonnegative signals.

### 1.3.1 Recovery thresholds of sparse recovery techniques

One natural way to recover a sparse vector from an underdetermined linear system is to find the sparsest solution among all the feasible ones, and it can be formulated into an $\ell_0$-minimization problem. Solving $\ell_0$-minimization in general is computationally hard. Its convexified version – $\ell_1$-minimization, however, can be solved efficiently and has proven recovery performance guarantee. People generally believe that as $p$ decreases from one to zero, the recovery performance of $\ell_p$-minimization should improve, i.e., the recovery threshold should increase, despite the fact that for every $p$ less than one, $\ell_p$-minimization is still computationally hard. Then, *what exactly is the recovery threshold of $\ell_p$-minimization ($p \in [0, 1)$)? Does $\ell_p$-minimization ($p \in [0, 1)$) really outperform $\ell_1$-minimization in sparse recovery?*

In Chapter 2, we characterize the recovery threshold of $\ell_p$-minimization ($p \in [0, 1)$). We consider the performance of strong recovery where all the vectors up to a certain sparsity should be correctly recovered. Moreover, we for the first time analyze the weak recovery performance of $\ell_p$-minimization where we need to recover all the sparse vectors on one support with one sign pattern. The strong recovery threshold increases when $p$ decreases from one to zero, which coincides with the conventional intuition. Surprisingly, the weak recovery threshold of $\ell_p$-minimization for all $p \in [0, 1)$ is *lower* than that of $\ell_1$-minimization. In this case, the weak recovery performance of $\ell_p$-minimization for all $p \in [0, 1)$ is *NOT comparable* to that of $\ell_1$-minimization, even though the former one is much harder to solve.

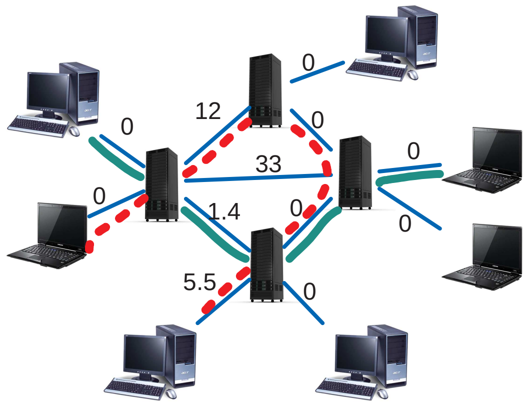



Figure 1.1: Network Example

### 1.3.2 Measurement constructions with topological constraints

In compressed sensing, an explicit construction of measurement matrices with a small number of measurements is still an open problem. Furthermore, current measurement constructions in compressed sensing critically rely on the assumption that a measurement can be a linear combination of any subset of the entries of an unknown vector. In problems such as network monitoring, however, the sum of certain entries of a vector may not be aggregated in one measurement due to topological constraints. For example, in Fig. 1.1, transmission delay occurs at a small number of bottle network links, and there is no delay on most links. We want to infer these link delays from a small number of end-to-end path delay measurements. The path in red dashed line and the path in green solid line are both valid paths in this network, and we can measure the corresponding end-to-end path delays. However, we can not measure the sum of delays on certain links if they do not form a valid path. Then, *how can we design measurements as few as possible so as to recover sparse vectors in the presence of network topological constraints?*

We address this issue in Chapter 3. We employ a graph to capture the topological constraints. Each node in the graph represents an entry of a vector, and a subset of nodes can be measured together only if they can induce a connected subgraph. Explicit measurement constructions for various graphs are provided, and the number of the constructed measurements is less than the existing estimate of the number of measurements required to recover sparse vectors over graphs. For general graphs, we also propose a design guideline for measurement constructions and further design a measurement construction algorithm.

### 1.3.3 A unique nonnegative solution to an underdetermined linear system

In many engineering applications, e.g. network monitoring, the sparse vector to recover is nonnegative, like the link delays. Recent studies [16, 56, 85] suggest that if an underdetermined linear system has a sparse nonnegative solution, then that solution is the only nonnegative solution to the system. Since the feasible set is indeed a singleton, this interesting phenomenon can potentially lead to efficient recovery techniques. *Under what conditions is a sparse nonnegative vector guaranteed to be the unique nonnegative solution?* We prove the existence of the singleton property for 0-1 measurement matrices in Chapter 4. We further prove that a sparse nonnegative vector can be the unique nonnegative solution to an underdetermined linear system even though the number of positive entries is proportional to the dimension.

CHAPTER 2

# SPARSE RECOVERY VIA $\ell_P$-MINIMIZATION ($0 \leq P \leq 1$)

Compressed sensing theory indicates that a high-dimensional sparse vector $\mathbf{x}^*$ in $\mathcal{R}^n$ can be recovered from low-dimensional measurements $\mathbf{y} = A\mathbf{x}^*$. In this chapter, with $A$ being a random Gaussian matrix, we investigate the recovering ability of $\ell_p$-minimization ($0 \leq p \leq 1$), where $\ell_p$-minimization returns a vector with the least $\ell_p$ quasi-norm among all the vectors $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{y}$. Besides analyzing the performance of strong recovery where $\ell_p$-minimization is required to recover all the sparse vectors up to certain sparsity, we also for the first time analyze the performance of "weak" recovery of $\ell_p$-minimization ($0 \leq p < 1$) where the aim is to recover all the sparse vectors on one support with a fixed sign pattern.

## 2.1 Introduction

Compressed sensing considers recovering a sparse vector $\mathbf{x}$ in $\mathcal{R}^n$ from an $m$-dimensional measurement $\mathbf{y} = A\mathbf{x}$. As discussed in Section 1.1.2, one natural estimate is to solve the the $\ell_0$-minimization problem in (1.3), and we reproduce it here for convenience:

$$\min_{\mathbf{x} \in \mathcal{R}^n} \|\mathbf{x}\|_0 \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{y}.$$

Since $\ell_0$-minimization is computationally hard to solve, people usually solve the convexified $\ell_1$-minimization in (1.4) instead, i.e.,

$$\min_{\mathbf{x} \in \mathcal{R}^n} \|\mathbf{x}\|_1 \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{y},$$

Conditions under which $\ell_1$-minimization can successfully recover $\mathbf{x}$ have been extensively studied in the literature [20, 21, 47, 55].

Among the explosion of research on compressed sensing ([7, 8, 9, 16, 37, 77, 135, 137, 139]) recently, there has been great research interest in recovering $\mathbf{x}$ by $\ell_p$-minimization for $0 < p < 1$ ([26, 27, 30, 29, 42, 43, 68, 111]) as follows,

$$\min_{\mathbf{x} \in \mathcal{R}^n} \|\mathbf{x}\|_p \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{y}. \tag{2.1}$$

Recall that $\|\mathbf{x}\|_p^p := (\sum_i |x_i|^p)$ for $p > 0$. Though $\|\cdot\|_p$ is a quasi-norm when $p < 1$ as it violates the triangular inequality, $\|\cdot\|_p^p$ follows the triangular inequality. We say $\mathbf{x}$ can be recovered by $\ell_p$-minimization if and only if it is the unique solution to (2.1). (2.1) is non-convex, and finding the global minimum is in general computationally hard. Chartrand [26, 27], Chartrand and Yin [29] employ heuristic algorithms to compute a local minimum of (2.1) and show numerically that these heuristics can indeed recover sparse vectors, and the support size of these vectors can be larger than that of the vectors recoverable from $\ell_1$-minimization. Then the question is what is the relationship between the sparsity of a vector and the successful recovery with $\ell_p$-minimization ($p < 1$)? How sparse should a vector be so that $\ell_p$-minimization can recover it? What is the threshold of sparsity that differentiates the success and failure of recovering by $\ell_p$-minimization? Gribonval and Nielsen [73] showed the sparsity up to which $\ell_p$-minimization can successfully recover all the sparse vectors at least does not decrease as $p$ decreases. Saab *et al.* [111] provided a sufficient condition for successful recovery via $\ell_p$-minimization based on Restricted Isometry Constants and provided a lower bound of the support size up to which $\ell_p$-minimization can recover all such sparse vectors. Chartrand and Staneva [30] improved this result by considering a modified Restricted $p$-Isometry Constant. Foucart and Lai [68] provided a lower bound of recovery threshold by considering a generalized version of RIP condition, and Blanchard *et al.*[11] numerically calculated this bound.

Our main contributions are as follows. For strong recovery where $\ell_p$-

minimization needs to recover all the vectors up to a certain sparsity, we provide a sharp threshold $\rho^*(p)$ of the ratio of the support size to the dimension which differentiates the success and the failure of $\ell_p$-minimization when $\alpha(=\frac{m}{n}) \to 1$. This is an exact threshold compared with a lower bound of successful recovery in previous results. When $p$ increases from 0 to 1, $\rho^*(p)$ decreases from 0.5 to 0.239. This coincides with the intuition that the performance of $\ell_p$-minimization is improved when $p$ decreases. When $\alpha \in (0,1)$ is fixed, we provide a positive bound $\rho^*(\alpha, p)$ for all $\alpha \in (0,1)$ and all $p \in (0,1]$ of strong recovery such that with a Gaussian measurement matrix $A^{m\times n}$, $\ell_p$-minimization can recover all the $\rho^*(\alpha, p)n$-sparse vectors with overwhelming probability. $\rho^*(\alpha, p)$ improves on the existing bounds in large $\alpha$ region.

We also analyze the performance of $\ell_p$-minimization for *weak* recovery where we need to recover all the sparse vectors on one support with one sign pattern. To the best of our knowledge, there is no existing result in this regard for $p < 1$. We characterize the successful weak recovery through a necessary and sufficient condition regarding the null space of the measurement matrix. When $\alpha \to 1$, we provide a sharp threshold $\rho^*_w(p)$ of the ratio of the support size to the dimension which differentiates the success and the failure of $\ell_p$-minimization. The weak threshold indicates that if we would like to recover every vector over one support with size less than $\rho^*_w(p)n$ and with one sign pattern, (though the support and sign patterns are not known a priori), and we generate a random Gaussian measurement matrix independently of the vectors, then with overwhelmingly high probability, $\ell_p$-minimization will recover all such vectors regardless of the amplitudes of the entries of a vector. For $\ell_1$-minimization, given a vector, if we randomly generate a Gaussian matrix and apply $\ell_1$-minimization, then its recovering ability observed in simulation exactly captures the weak recovery threshold, see [53, 55]. Interestingly, when $\alpha \to 1$ and $n$ is large enough, we prove that the weak threshold $\rho_w^*(p)$ is $2/3$ for all $p \in [0, 1)$, and is lower than the weak threshold of $\ell_1$-minimization, which is 1. In this region, $\ell_1$-minimization outperforms $\ell_p$-minimization for all $p \in [0, 1)$ if we only need to recover sparse vectors on one support with one sign pattern. We also explicitly show that $\ell_p$-minimization ($p \in (0, 1)$) can return a vector denser than the original sparse vector while $\ell_1$-minimization successfully recovers the sparse vector. Finally, for every $\alpha \in (0, 1)$, we provide a positive bound $\rho_w^*(\alpha, p)$ such that $\ell_p$-minimization successfully recovers all the $\rho_w^*(\alpha, p)n$-sparse vectors on one support with one sign pattern.

The rest of this chapter is organized as follows. We introduce the null space condition of successful $\ell_p$-minimization in Section 2.2. We especially define the successful weak recovery for $p < 1$ and provide a necessary and sufficient condition. We use an example to illustrate that the solution of $\ell_1$-minimization can be sparser than that of $\ell_p$-minimization ($p \in (0, 1)$). Section 2.3 provides thresholds of the sparsity ratio of the successful recovery via $\ell_p$-minimization for all $p \in [0, 1]$ both in strong recovery and in weak recovery when the measurement matrix is random Gaussian matrix and $\alpha \to 1$. For $\alpha \in (0, 1)$, Section 2.4 provides bounds of sparsity ratio below which $\ell_p$-minimization is successful in the strong sense and in the weak sense respectively. We compare the performance of $\ell_p$-minimization ($p < 1$) and the performance of $\ell_1$-minimization in Section 2.5 and provide numerical results in Section 2.6. We only state the results in the main text and please refer to the Appendix for the proofs.

## 2.2 Successful Recovery of $\ell_p$-minimization

We first introduce the null space characterization of the measurement matrix $A$ to capture the successful recovery via $\ell_p$-minimization ($p \in [0,1]$). Besides the strong recovery that has been studied in [11, 37, 68, 69, 73, 111, 119], we especially provide a necessary and sufficient condition for the success of *weak* recovery in the sense that $\ell_p$-minimization only needs to recover all the sparse vectors on one support with one sign pattern. For example, in practice, given an unknown vector to recover, we randomly generate a measurement matrix and solve the $\ell_1$-minimization problem, the simulation result of recovery performance with respect to the sparsity of the vector indeed represents the performance of weak recovery.

Given a measurement matrix $A^{m \times n}$, let $B^{n \times (n-m)}$ denote a matrix whose columns form a basis of the null space of $A$, then we have $AB = \mathbf{0}$. Let $B_i$ ($i \in \{1, ..., n\}$) denote the $i^{\text{th}}$ row of $B$. Let $B_T$ denote the submatrix of $B$ with $T \subseteq \{1, ..., n\}$ as the set of row indices. Let $T^c \subseteq \{1, ..., n\}$ be the complimentary set of $T$. In this chapter, we will study the sparse recovery property of $\ell_p$-minimization by analyzing the null space of $A$.

We first state the null space condition for the success of strong recovery via $\ell_p$-minimization [63, 73] in the sense that $\ell_p$-minimization should recover all the sparse vectors up to a certain sparsity.

**Theorem 1** ([63, 73]). $\mathbf{x}$ *is the unique solution to $\ell_p$-minimization problem* ($0 \leq p \leq 1$) *for every vector* $\mathbf{x}$ *up to $\rho n$-sparse if and only if*

$$\|B_T \mathbf{z}\|_p^p < \|B_{T^c} \mathbf{z}\|_p^p \tag{2.2}$$

*for every non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$, and every support $T$ with $|T| \leq \rho n$.*

One important property is that if the condition (2.2) is satisfied for some $0 < p \leq 1$, then it is also satisfied for all $q \in [0, p]$ [43, 74]. Therefore, if $\ell_p$-minimization could recover all the $\rho n$-sparse vectors $\mathbf{x}$, then $\ell_q$-minimization ($0 \leq q \leq p$) could also recover all the $\rho n$-sparse vectors. Intuitively, the strong recovery performance of $\ell_q$-minimization should be at least as good as that of $\ell_p$-minimization when $0 \leq q < p \leq 1$.

### 2.2.1 Weak recovery for $\ell_p$-minimization

Though $\ell_p$-minimization ($p < 1$) should be at least as good as $\ell_1$-minimization for strong recovery, the argument may not be true for weak recovery. For weak recovery, we would like to recover all the vectors on some support $T$ with some sign pattern $\sigma$, and $\sigma_i \in \{1, -1\}$ for every $i$ in $T$. $\sigma_i = 1$ if a vector is positive on index $i$, and $\sigma_i = -1$ if a vector is negative on index $i$. Given any non-zero vector $\mathbf{z} \in \mathcal{R}^{n-m}$, we define $T^- := \{i \in T : B_i\mathbf{z}\sigma_i < 0\}$, $T^+ := \{i \in T : B_i\mathbf{z}\sigma_i > 0\}$, and $T^0 := \{i \in T : B_i\mathbf{z} = 0\}$. Note that when $B$ is given, $T^-$, $T^+$ and $T^0$ depend on $\mathbf{z}$, and they can be empty. In this dissertation for weak recovery, we consider recovering nonnegative vectors on some support $T$ for notational simplicity. In this case, $T^-$ and $T^+$ are simplified to be $T^- = \{i \in T : B_i\mathbf{z} < 0\}$ and $T^+ = \{i \in T : B_i\mathbf{z} > 0\}$. However, all the results also hold for any specific support and any sign pattern.

We first state the null space condition for successful weak recovery via $\ell_1$-minimization as follows, please see [49, 73, 118, 140, 144] for this result.

**Theorem 2.** *For every nonnegative $\mathbf{x} \in \mathcal{R}^n$ on some support $T$, $\mathbf{x}$ is the unique solution* *to $\ell_1$-minimization problem (1.4) if and only if*

$$\|B_{T^-}\mathbf{z}\|_1 < \|B_{T^c}\mathbf{z}\|_1 + \|B_{T^+}\mathbf{z}\|_1 \tag{2.3}$$

*holds for all non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$.*

Note that for every nonnegative vector $\mathbf{x}$ on a fixed support $T$, the condition to successfully recover it via $\ell_1$-minimization is the same, as stated in Theorem 2. Therefore if one vector $\mathbf{x}$ can be successfully recovered, all the other nonnegative sparse vectors on $T$ can also be recovered. Conversely, if some vector $\mathbf{x}$ cannot be successfully recovered, then every other nonnegative vector on $T$ cannot be recovered either. However, the condition of successful recovery via $\ell_p$-minimization ($0 \leq p < 1$) varies for different nonnegative sparse vectors even if they have the same support. In other words, the recovery condition depends on the amplitudes of the entries of the vector. Here we consider the worst case scenario for weak recovery in the sense that the recovery via $\ell_p$-minimization is defined to be "successful" if it can recover *all* the nonnegative vectors on a fixed support. Mathematically, we have

**Definition 1** (Weak recovery of $\ell_p$-minimization)**.** *Given $p \in [0, 1]$ and support $T$, if it holds that $\mathbf{x}$ is the unique solution to $\ell_p$-minimization problem for all nonnegative vectors $\mathbf{x} \in \mathcal{R}^n$ on $T$, then we say the weak recovery of $\ell_p$-minimization is successful in respect to nonnegative vectors on $T$.*

Under Definition 1, the null space condition for weak recovery of $\ell_1$-minimization is still the same as that in Theorem 2. We characterize the $\ell_p$-minimization ($p \in (0, 1)$) case in Theorem 3 and the $\ell_0$-minimization case in Theorem 4.

**Theorem 3.** *Given any $p \in (0,1)$, the weak recovery of $\ell_p$-minimization (2.1) is successful respect to nonnegative vectors on support $T$, if and only if the following condition holds for every non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$:*

*if $T^+$ is not empty, then*

$$||B_{T^-}\mathbf{z}||_p^p \leq ||B_{T^c}\mathbf{z}||_p^p; \tag{2.4}$$

*and if $T^+$ is empty, then*

$$||B_{T^-}\mathbf{z}||_p^p < ||B_{T^c}\mathbf{z}||_p^p.$$

Similarly, the null space condition for the weak recovery of $\ell_0$-minimization is as follows, we skip its proof as it is similar to that of Theorem 3.

**Theorem 4.** *The weak recovery of $\ell_0$-minimization (1.3) is successful in respect to nonnegative vectors on support $T$, if and only if*

$$||B_{T^-}\mathbf{z}||_0 < ||B_{T^c}\mathbf{z}||_0 \tag{2.5}$$

*for all non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$.*

For the strong recovery, the null space conditions of $\ell_1$-minimization and $\ell_p$-minimization ($0 \leq p < 1$) share the same form (2.2), and if (2.2) holds for some $p \leq 1$, it also holds for all $q \in [0, p]$. However, for recovery of sparse vectors on one support with one sign pattern, from Theorem 2, 3 and 4, we know that although the conditions of $\ell_p$-minimization ($0 < p < 1$) and $\ell_0$-minimization share a similar form in (2.4) and (2.5), the condition of $\ell_1$-minimization has a very different form in (2.3). Moreover, if (2.4) holds for some $p \in (0,1)$, it does not necessarily hold for all $q \in (0, p)$. Therefore the way that the performance of weak recovery changes over $p$ may be quite different from the way that the performance of strong recovery changes over $p$. Moreover, the performance of weak recovery of $\ell_1$ may be significantly different from that of $\ell_p$-minimization for $p \in (0,1)$. We will further discuss this issue.

### 2.2.2 The solution of $\ell_1$-minimization can be sparser than that of $\ell_p$-minimization ($p \in (0,1)$)

$\ell_p$-minimization ($p \in (0,1)$) may not perform as well as $\ell_1$-minimization in some cases, for example in the weak recovery which we will discuss in Section 2.3 and Section 2.4. Here we employ a numerical example to illustrate that in certain cases $\ell_1$-minimization can recover the sparse vector while $\ell_p$-minimization ($p \in (0,1)$) cannot, and the solution of $\ell_p$-minimization is denser than the original sparse vector.

**Example 1.** $\ell_p$-minimization returns a denser solution than $\ell_1$-minimization.

Let the measurement matrix $A$ be a $(6k-1)\times 6k$ matrix with $\boldsymbol{\beta} \in \mathcal{R}^{6k}$ as a basis of its null space, and $\beta_i = 1$ for all $i \in \{1,...,k\}$, $\beta_i = -1$ for all $i \in \{k+1,...,2k\}$, and $\beta_i = 1/64$ for all $i \in \{2k+1,...,6k\}$. Then every vector in the null space can be represented as $h\beta$, for some $h \in \mathcal{R}$. Note that $\|h\beta\|_1/2 = \frac{33k|h|}{32}$, and $\|h\beta_T\|_1 \leq (\lceil\frac{33}{32}k\rceil - 1)|h| < \|h\beta\|_1/2$ for all $T \subset \{1,...,6k\}$ with $|T| \leq (\lceil\frac{33}{32}k\rceil - 1)$ and for all $h \in \mathcal{R}$, and $\|h\beta_{\hat{T}}\|_1 = \lceil\frac{33}{32}k\rceil|h| \geq \|h\beta\|_1/2$ for all $h$ if $\hat{T} = \{1,...,\lceil\frac{33}{32}k\rceil\}$. Then according to Theorem 1, $\ell_1$-minimization can recover all the $(\lceil\frac{33}{32}k\rceil - 1)$-sparse vectors in $\mathcal{R}^{6k}$, but fails to recover some $\lceil\frac{33}{32}k\rceil$-sparse vector. Similarly, $\|h\beta\|_{0.5}^{0.5}/2 = \frac{5k|h|}{4}$, and $\|h\beta_T\|_{0.5}^{0.5} \leq (\lceil\frac{5}{4}k\rceil - 1)|h| < \|h\beta\|_{0.5}^{0.5}/2$ for all $T \subset \{1,...,6k\}$ with $|T| \leq (\lceil\frac{5}{4}k\rceil - 1)$ and for all $h \in \mathcal{R}$, and $\|h\beta_{\hat{T}}\|_{0.5}^{0.5} = \lceil\frac{5}{4}k\rceil|h| \geq \|h\beta\|_{0.5}^{0.5}/2$ for all $h$ if $\hat{T} = \{1,...,\lceil\frac{5}{4}k\rceil\}$. Therefore by Theorem 1, $\ell_{0.5}$-minimization can recover all the $(\lceil\frac{5}{4}k\rceil - 1)$-sparse vectors in

$\mathcal{R}^{6k}$, but fails to recover some $\lceil \frac{5}{4}k \rceil$-sparse vector. Therefore, in terms of strong recovery, $\ell_{0.5}$-minimization has a better performance than $\ell_1$-minimization as it can recover all the vectors up to a higher sparsity.

Before discussing the weak recovery performance, we should first point out that when the null space is only one-dimensional, the $\ell_p$-minimization problem for all $p \in (0, 1]$ can be easily solved. Let $\mathbf{x}^*$ denote the sparse vector we would like to recover, and let $\tilde{\mathbf{x}}$ denote a vector that can produce the same measurements as $\mathbf{x}^*$, and mathematically, $A\tilde{\mathbf{x}} = A\mathbf{x}^*$. Then every vector $\mathbf{x}$ such that $A\mathbf{x} = A\mathbf{x}^*$ holds should satisfy $\mathbf{x} = \tilde{\mathbf{x}} + h\boldsymbol{\beta}$ for some $h \in \mathcal{R}$. Then the $\ell_p$-minimization problem ($p \in (0, 1]$) is equivalent to

$$\min_{h \in \mathcal{R}} \|\tilde{\mathbf{x}} + h\boldsymbol{\beta}\|_p^p. \tag{2.6}$$

Given $\tilde{\mathbf{x}}$ and $\boldsymbol{\beta}$, $\|\tilde{\mathbf{x}}+h\boldsymbol{\beta}\|_p^p$ is a function of $h$. Define set $S = \{-\frac{\tilde{x}_i}{\beta_i} \mid \beta_i \neq 0\}$, let $q$ denote the number of different elements in $S$, and let $s_i$ $(i = 1, ..., q)$ denote the ordered elements in $S$, and $s_i < s_j$ if $i < j$. Let $I_0$ denote the interval $(-\infty, s_1]$, let $I_i$ denote the interval $[s_i, s_{i+1}]$ $(i = 1, ..., q-1)$, and let $I_q$ denote the interval $[s_q, +\infty)$. Note that for each interval $I_i$ $(0 \leq i \leq q)$, $\|\tilde{\mathbf{x}} + h\boldsymbol{\beta}\|_p^p$ is concave on $I_i$ for every $p \in (0, 1)$, and $\|\tilde{\mathbf{x}} + h\boldsymbol{\beta}\|_1$ is linear on $I_i$. Therefore the minimum value of $\|\tilde{\mathbf{x}} + h\boldsymbol{\beta}\|_p^p$ $(p \in (0, 1])$ on $I_i$ $(1 \leq i \leq q-1)$ should be achieved at one of the endpoints of $I_i$, either $s_i$ or $s_{i+1}$. Since when $h$ goes to $-\infty$ or $+\infty$, $\|\tilde{\mathbf{x}} + h\boldsymbol{\beta}\|_p^p$ goes to $+\infty$, then the minimum value of $\|\tilde{\mathbf{x}} + h\boldsymbol{\beta}\|_p^p$ $(p \in (0, 1])$ on $I_1$ should be achieved at $s_1$, and the minimum value on $I_{q+1}$ should be achieved at $s_q$. Thus, let $\mathbf{x}^i = \tilde{\mathbf{x}} + s_i\boldsymbol{\beta}$ for every $i = 1, ..., q$, and let $i^* := \arg\min_{1 \leq i \leq q} \|\mathbf{x}^i\|_p^p$, then $\mathbf{x}^{i^*}$ is the solution to $\ell_p$-minimization problem. We call $\mathbf{x}^i$'s as "singular vectors". Therefore, to solve (2.6), we only need to find all the singular vectors, and the one with the least $\ell_p$ quasi-norm (or $\ell_1$ norm) is the solution to $\ell_p$-minimization (or $\ell_1$-minimization). If $\mathbf{x}^{i^*} = \mathbf{x}^*$, then we say $\mathbf{x}^*$ can be successfully recovered.

Now consider the "weak" recovery as to recover all the nonnegative vectors on support $T = \{1, ..., 2k\}$. According to Theorem 2 and Theorem 3, one can check that $\ell_1$-minimization can indeed recover all the nonnegative vectors on support $T$, however, $\ell_{0.5}$-minimization fails to recover some vectors in this case. For example, consider a $2k$-sparse vector $\mathbf{x}^*$ with $x_i^* = 9$ for all $i \in \{1, ..., k\}$, $x_i^* = 1$ for all $i \in \{k+1, ..., 2k\}$, and $x_i^* = 0$ for all $i \in \{2k+1, ..., 6k\}$. There are three singular vectors in this case: $\mathbf{x}^1 = \mathbf{x}^*$, $\mathbf{x}^2 = \mathbf{x}^* + \beta$ and $\mathbf{x}^3 = \mathbf{x}^* - 9\beta$. Since $\|\mathbf{x}^1\|_1 = 10k$, $\|\mathbf{x}^2\|_1 = 10k + k/16$, and $\|\mathbf{x}^3\|_1 = 10k + 9k/16$, then $\mathbf{x}^1$ is the solution of $\ell_1$-minimization, and $\mathbf{x}^*$ is successfully recovered. Now consider $\ell_{0.5}$-minimization, since $\|\mathbf{x}^1\|_{0.5}^{0.5} = 4k$, $\|\mathbf{x}^2\|_{0.5}^{0.5} = (\sqrt{10} + 0.5)k$, and $\|\mathbf{x}^3\|_{0.5}^{0.5} = (\sqrt{10} + 1.5)k$, then $\mathbf{x}^2$ is the solution of $\ell_{0.5}$-minimization, and it is $5k$-sparse. Thus, the solution of $\ell_{0.5}$-minimization is a $5k$-sparse vector although the original vector $\mathbf{x}^*$ is only $2k$-sparse. Therefore $\ell_{0.5}$-minimization fails to recover some nonnegative $2k$-sparse vector $\mathbf{x}^*$ while $\mathbf{x}^*$ is the solution to $\ell_1$-minimization, and the solution of $\ell_{0.5}$-minimization is denser than the original vector $\mathbf{x}^*$.

## 2.3 Recovery thresholds when $\frac{m}{n} \to 1$

In this chapter we focus on the case that the measurement matrix $A$ has i.i.d. standard Gaussian $\mathcal{N}(0, 1)$ entries. Then for a matrix $B^{n\times(n-m)}$ with i.i.d. $\mathcal{N}(0, 1)$ entries, the column space of $B$ is equivalent in distribution to the null space of $A$, please refer to [21, 141] for details. Then in later analysis, we will use $B$ to represent a basis of the null space of $A$.

We first focus on the case that $\alpha = \frac{m}{n} \to 1$ and provide recovery thresholds of $\ell_p$-minimization for every $p \in [0, 1]$. We consider two types of thresholds:

one in the *strong* sense as we require $\ell_p$-minimization to recover *all* $\rho n$-sparse vectors (Section 2.3.1), one in the *weak* sense as we only require $\ell_p$-minimization to recover *all the vectors on a certain support with a certain sign pattern* (Section 2.3.2). Since in our setup the measurement matrix $A$ has i.i.d. $\mathcal{N}(0,1)$ entries, the weak recovery performance does not depend on the specific choice of the support and the sign pattern. We call it a threshold as for any sparsity below that threshold, $\ell_p$-minimization can recover all the sparse vectors either in the strong sense or the weak sense when $\alpha$ is close enough to 1 and $n$ is large enough, and for any sparsity above that threshold, $\ell_p$-minimization fails to recover some sparse vector no matter how large $\alpha$ and $n$ are. These thresholds can be viewed as the limiting behavior of $\ell_p$-minimization, since for any constant $\alpha \in (0,1)$, the recovery thresholds of $\ell_p$-minimization would be no greater than the ones provided here.

### 2.3.1 Strong Recovery

In this section, for given $p$, we shall provide a threshold $\rho^*(p)$ of *strong recovery* such that for any $\rho < \rho^*(p)$, $\ell_p$-minimization (2.1) can recover *all* $\rho n$-sparse vectors $\mathbf{x}$ with overwhelming probability when $\alpha$ is close enough to 1. Our technique here stems from [62], which only focuses on the strong recovery of $\ell_1$-minimization.

We have already discussed in Section 2.2 that the performance of $\ell_q$-minimization should be no worse than $\ell_p$-minimization for strong recovery when $0 \leq q < p \leq 1$. Although there are results about bound of the sparsity below which $\ell_p$-minimization can recover all the sparse vectors, no existing re-

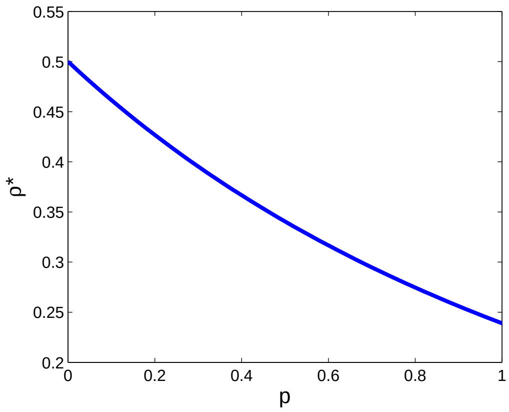



Figure 2.1: Threshold $\rho^*$ of successful recovery with $\ell_p$-minimization

sult has explicitly calculated the recovery threshold of $\ell_p$-minimization for $p < 1$ which differentiates the success and failure of $\ell_p$-minimization. To this end, we will first define $\rho^*(p)$ in the following lemma, and then prove that $\rho^*(p)$ is indeed the threshold of strong recovery in later part.

**Lemma 1.** *Let $X_1$, $X_2$,...,$X_n$ be i.i.d. $\mathcal{N}(0,1)$ random variables and let $Y_1$, $Y_2$,...,$Y_n$ be the sorted ordering (in non-increasing order) of $|X_1|^p$, $|X_2|^p$,...,$|X_n|^p$ for some $p \in (0,1]$. For given $\rho > 0$, define $S_\rho$ as $\sum_{i=1}^{\lceil \rho n \rceil} Y_i$. Let $S$ denote $E[S_1]$, the expected value of $S_1$. Then there exists a constant $\rho^*(p)$ such that $\lim_{n\to\infty} \frac{E[S_{\rho^*}]}{S} = \frac{1}{2}$.*

$\rho^*$ is a function of $p$, and in fact is strictly decreasing as stated in Proposition 1.

**Proposition 1.** *The function $\rho^*(p)$ is strictly decreasing in $p$ on $(0,1]$.*

Note that $\rho^*(p)$ goes to $\frac{1}{2}$ as $p$ tends to zero from (A.4) and (A.5). We plot $\rho^*$ against $p$ numerically in Fig. 2.1. We also obtain that $\rho^*(1) = 0.239...$, which coincides with the result in [62].

Now we proceed to prove that $\rho^*$ is the threshold of successful recovery with

$\ell_p$-minimization for $p$ in $(0, 1]$. First we state the concentration property of $S_\rho$ in the following lemma.

**Lemma 2.** *For any $p \in (0, 1]$, let $X_1$,...,$X_n$, $Y_1$,...,$Y_n$, $S_\rho$ and $S$ be as in Lemma 1. For any $\rho > 0$ and any $\delta > 0$, there exists a constant $c_1 > 0$ such that when n is large enough, with probability at least $1 - 2e^{-c_1 n}$, $|S_\rho - E[S_\rho]| \leq \delta S$.*

Roughly speaking, Lemma 2 states that $S_\rho$ is concentrated around its expectation $E[S_\rho]$ for every $\rho$. For our purpose in this dissertation, the following two corollaries of Lemma 2 are important for the later proof.

**Corollary 1.** *For any $\rho < \rho^*$, there exists a $\delta > 0$ and a constant $c_2 > 0$ such that when n is large enough, with probability at least $1 - 2e^{-c_2 n}$, $S_\rho \leq (\frac{1}{2} - \delta)S$.*

**Corollary 2.** *For any $\epsilon > 0$, there exists a constant $c_3 > 0$ such that when n is large enough, with probability at least $1 - 2e^{-c_3 n}$, it holds that $(1 - \epsilon)S \leq S_1 \leq (1 + \epsilon)S$.*

From the above two corollaries and applying the union bound, one can easily show that with overwhelming probability the sum of the largest $\lceil \rho n \rceil$ terms of $Y_i$'s is less than half of the total sum $S_1$ if $\rho < \rho^*$. The following lemma extends the result to all the vectors $B\mathbf{z}$ simultaneously where matrix $B^{n\times(n-m)}$ has i.i.d. Gaussian entries and $\mathbf{z}$ is any non-zero vector in $\mathcal{R}^{n-m}$.

**Lemma 3.** *For any $0 < p \leq 1$, given any $\rho < \rho^*(p)$, there exist constants $0 < c_4 < 1$, $c_5 > 0$, $\delta > 0$ such that when $\alpha = \frac{m}{n} > c_4$ and n is large enough, with probability at least $1 - e^{-c_5 n}$, an $n \times (n - m)$ matrix $B$ with i.i.d. $\mathcal{N}(0, 1)$ entries has the following property: for every non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$ and every subset $T \subseteq \{1, ..., n\}$ with $|T| \leq \lceil \rho n \rceil$, $\|B_{T^c}\mathbf{z}\|_p^p - \|B_T\mathbf{z}\|_p^p \geq \delta S \|\mathbf{z}\|_2^p$.*

We remark here that in Lemma 3 and all the following results in this dissertation, when we say "with probability at least $1 - e^{-cn}$ for some constant $c > 0$", by “constant” we mean $c$ does not depend on the measurement matrix $A$, but $c$ could depend on other parameters in various occasions.

Lemma 3 indicates that when $\alpha > c_4$ and $n$ is large enough, with overwhelming probability $\sum_{i\in T^c} |(B\mathbf{z})_i|^p - \sum_{i\in T} |(B\mathbf{z})_i|^p \geq \delta S \,\|\mathbf{z}\|_2^p > 0$ holds for every non-zero $\mathbf{z}$ and every set $T$ with $|T| \leq \lceil \rho n \rceil$, then from Theorem 1, in this case every $\lceil \rho n \rceil$-sparse vector $\mathbf{x}$ is the unique solution to the $\ell_p$-minimization problem (2.1) with overwhelming probability. We can now establish one main result regarding the threshold of successful recovery via $\ell_p$-minimization.

**Theorem 5.** *For any $0 < p \leq 1$, given any $\rho < \rho^*(p)$, there exist constants $0 < c_4 < 1$, $c_5 > 0$ such that when $\alpha > c_4$ and n is large enough, with probability at least $1 - e^{-c_5 n}$, an $m \times n$ matrix A with i.i.d. $\mathcal{N}(0,1)$ entries has the following property: for every $\mathbf{x} \in \mathcal{R}^n$ with its support T satisfying $|T| \leq \lceil \rho n \rceil$, $\mathbf{x}$ is the unique solution to the $\ell_p$-minimization problem (2.1).*

We remark here that $\rho^*(p)$ is a sharp bound for successful recovery. For any $\rho > \rho^*(p)$, from Lemma 1 and Lemma 2, for any $\mathbf{z}$ in $\mathcal{R}^{n-m}$, with overwhelming probability the sum of the largest $\lceil \rho n \rceil$ terms of $|B_i\mathbf{z}|^p$’s is more than the half of the total sum $S_1$, i.e. the null space condition stated in Theorem 1 for successful recovery via $\ell_p$-minimization fails with overwhelming probability. Therefore, $\ell_p$-minimization fails to recover some $\rho n$-sparse vector with overwhelming probability if $\rho > \rho^*(p)$. Proposition 1 implies that the threshold strictly decreases as $p$ increases. The performance of $\ell_{p_1}$-minimization is better than that of $\ell_{p_2}$-minimization for $0 < p_1 < p_2 \leq 1$ in strong recovery as $\ell_{p_1}$-minimization can recover vectors up to a higher sparsity.

### 2.3.2 Weak Recovery

We have demonstrated in Section 2.3.1 that the threshold for strong recovery strictly decreases as $p$ increases from 0 to 1. Here we provide a weak recovery threshold for all $p \in [0, 1)$ when $\alpha \to 1$. As we shall see, for weak recovery, the threshold of $\ell_p$-minimization is the same for all $p \in [0, 1)$, and is lower than the threshold of $\ell_1$-minimization.

Recall that for successful weak recovery, $\ell_p$-minimization should recover all the vectors on some fixed support with a fixed sign pattern, and the equivalent null space characterization is stated in Theorem 3 and Theorem 4.

Note that to simply the notation, for the remaining part of this chapter, we will say a vector is $\rho n$-sparse or the size of the support is $\rho n$ instead of using the notation $\lceil \rho n \rceil$. However, the support size should always be an integer.

We define $x^0 = 1$ for all $x \neq 0$, and $0^0 = 0$. To characterize the recovery threshold of $\ell_p$-minimization in this case, we first state the following lemma.

**Lemma 4.** *Let $X_1$, $X_2$,...,$X_n$ be i.i.d. $\mathcal{N}(0, 1)$ random variables and $T$ be a set of indices with size $|T| = \rho n$ for some $\rho > 0$. For every $p \in [0, 1)$, for every $\epsilon > 0$, when $n$ is large enough, with probability at least $1 - e^{-c_6 n}$ for some constant $c_6 > 0$, the following two properties hold simultaneously:*

- $\frac{1}{2}\rho n(\mu - \epsilon) < \sum_{i \in T : X_i < 0} |X_i|^p < \frac{1}{2}\rho n(\mu + \epsilon)$
- $(1 - \rho)n(\mu - \epsilon) < \sum_{i \in T^c} |X_i|^p < (1 - \rho)n(\mu + \epsilon)$.

*where $\mu = E[|X|^p]$, $X \sim \mathcal{N}(0, 1)$.*

The proof of Lemma 4 is based on concentration of measure, and the arguments are similar to those in the proof of Lemma 2. Lemma 4 implies that $\sum_{i \in T: X_i < 0} |X_i|^p < \sum_{i \in T^c} |X_i|^p$ holds with high probability when $|T| = \rho n < \frac{2}{3}n$. Applying the net arguments similar to those in the proof of Lemma 3, we can also show that with overwhelming probability the statement holds for all vectors $B\mathbf{z}$ simultaneously where matrix $B^{n \times (n-m)}$ has i.i.d. Gaussian entries and $\mathbf{z}$ is any non-zero vector in $\mathcal{R}^{n-m}$. Then we can establish the main result regarding the threshold of successful recovery with $\ell_p$-minimization from vectors on one support with the same sign pattern.

**Theorem 6.** *For any $p \in [0, 1)$, given any $\rho < \rho_w^* := \frac{2}{3}$, there exist constants $c_7 \in (0, 1)$, $c_8 > 0$ such that when $\alpha > c_7$ and $n$ is large enough, with probability at least $1 - e^{-c_8 n}$, an $m \times n$ matrix $A$ with i.i.d. $\mathcal{N}(0, 1)$ entries has the following property: for every nonnegative vector $\mathbf{x}$ on some support $T$ satisfying $|T| \leq \rho n$, $\mathbf{x}$ is the unique solution to the $\ell_p$-minimization problem.*

We remark here that $\rho_w^*$ is a sharp bound for successful recovery in this setup. For any $\rho > \rho_w^*$, from Lemma 4, with overwhelming probability that $\sum_{i \in T: B_i\mathbf{z} < 0} |B_i\mathbf{z}|^p > \sum_{i \in T^c} |B_i\mathbf{z}|^p$, then Theorem 3 and Theorem 4 indicate that the $\ell_p$-minimization ($p \in [0, 1)$) fails to recover some nonnegative $\rho n$-sparse vector $\mathbf{x}$ on $T$ in this case. Note that for a random Gaussian measurement matrix, from symmetry one can check that this result does not depend on the specific choice of the support and the sign pattern. In fact, $\rho_w^*$ in Theorem 6 is the weak recovery threshold for any fixed support and any fixed sign pattern.

Surprisingly, the successful recovery threshold $\rho_w^*$ when we only consider recovering vectors on one support with one sign pattern is $\frac{2}{3}$ for all $p$ in $[0, 1)$ and is strictly less than the threshold for $p = 1$, which is 1 [53]. Thus in this case, $\ell_1$-minimization has better recovery performance than $\ell_p$-minimization ($p \in [0, 1)$) in terms of the sparsity requirement for the sparse vector. Although the strong recovery performance can be improved if we apply $\ell_p$-minimization with a smaller $p$, $\ell_1$-minimization can indeed outperform $\ell_p$-minimization for all $p \in [0, 1)$ in weak recovery if $\alpha$ is close to 1 and $n$ is large enough.

It might be counterintuitive at first sight to see that the weak threshold of $\ell_0$-minimization is less than that of $\ell_1$-minimization, so let us take a moment to consider what the result means. We choose recovering all nonnegative vectors on some support $T$ ($|T| = \rho n$) for the weak recovery, the argument follows for all the other supports and all the other sign patterns. The results about weak recovery threshold indicate that for any $\rho \in (2/3, 1)$, when $n$ is sufficiently large and $\alpha$ is close enough to 1, for a random Gaussian measurement matrix $A$, $\ell_1$-minimization would recover all the nonnegative vectors on support $T$ with overwhelming probability, while $\ell_0$-minimization would fail to recover some nonnegative vector on $T$ with overwhelming probability. The failure of $\ell_0$-minimization indicates that there exists a nonnegative vector $\mathbf{x}$ on support $T$ and a vector $\mathbf{x}'$ on support $T'$ such that $|T'| \leq |T|$, and $A\mathbf{x} = A\mathbf{x}'$. Note that $\mathbf{x}'$ could have negative entries, or $T'$ may not be a subset of $T$. Therefore, if $\mathbf{x}$ is the sparse vector we would like to recover from $A\mathbf{x}$, $\ell_0$-minimization would fail since $\|\mathbf{x}'\|_0 \leq \|\mathbf{x}\|_0$. However, $\|\mathbf{x}\|_1 < \|\mathbf{x}'\|_1$ should hold since $\ell_1$-minimization can successfully return $\mathbf{x}$ as its solution. Of course when $\mathbf{x}'$ is the sparse vector we would like to recover, $\ell_1$-minimization would return $\mathbf{x}$ and fail to recover $\mathbf{x}'$. However, since $\ell_1$-minimization would recover all the nonnegative vectors on $T$, then either $T' \nsubseteq T$ holds or $\mathbf{x}'$ has negative entries. Therefore when we consider recovering nonnegative vectors on $T$ for the weak recovery, $\mathbf{x}'$ is not taken into account, and $\ell_1$-minimization works better than $\ell_0$-minimization.

Thus, although the performance of $\ell_1$-minimization is not as good as that of $\ell_p$-minimization ($p \in [0, 1)$) in the strong recovery which requires to recover all the vectors up to certain sparsity, $\ell_1$-minimization can recover all the $\rho n$-sparse ($\rho > 2/3$) vectors on some support with some sign pattern, while for $\ell_p$-minimization ($p \in [0, 1)$), the size of the largest support on which it can recover all the vectors with one sign pattern is no greater than $2n/3$. In a word, when we aim to recover all the vectors up to certain sparsity, $\ell_p$-minimization is better for smaller $p$, however, when we aim to recover all the vectors on one support with one sign pattern, $\ell_1$-minimization may have a better performance.

## 2.4 Recovery Bounds for Fixed $\frac{m}{n}$

We considered the limiting case that $\alpha \to 1$ in Section 2.3 and provided the limiting thresholds of sparsity ratio for successful recovery via $\ell_p$-minimization both in the strong sense and in the weak sense. Here we focus on the case that $\alpha$ is fixed ($0 < \alpha < 1$). For any $\alpha$ and $p$, we will provide a bound $\rho^*(\alpha, p)$ for strong recovery and a bound $\rho_w^*(\alpha, p)$ for weak recovery such that $\ell_p$-minimization can recover all the $\rho^*(\alpha, p)n$-sparse vectors with overwhelming probability, and recover all the $\rho_w^*(\alpha, p)n$-sparse vectors on one support with one sign pattern with overwhelming probability. Note that the thresholds we provided in Section 2.3 is tight in the sense that for any $\rho > \rho^*$ in the strong recovery or any $\rho > \rho_w^*$ in the weak recovery, with overwhelming probability $\ell_p$-minimization would fail to recover some $\rho n$-sparse vector. However, $\rho^*(\alpha, p)$ and $\rho_w^*(\alpha, p)$ we provide in this section are lower bounds for the thresholds of strong recovery and weak recovery respectively, and might not be tight in general.

### 2.4.1 Strong Recovery

From Theorem 1 we know that in order to successfully recover all the $\rho n$-sparse vectors via $\ell_p$-minimization, $\|B_T\mathbf{z}\|_p^p < \frac{1}{2}\|B\mathbf{z}\|_p^p$ should hold for every non-zero vector $\mathbf{z} \in \mathcal{R}^{n-m}$, and every set $T \subset \{1, ..., n\}$ with $|T| \le \rho n$. The key idea to obtain a lower bound $\rho^*(\alpha, p)$ is as follows. We first calculate a lower bound of $\|B\mathbf{z}\|_p^p$ for all $\mathbf{z}$ in $\mathcal{S}$, where $\mathcal{S}$ is the unit sphere in $\mathcal{R}^{n-m}$. Then for any $\rho$, we calculate an upper bound of $\|B_T\mathbf{z}\|_p^p$ for all $T$ with $|T| = \rho n$ and all $\mathbf{z}$ in $\mathcal{S}$. Then we define $\rho^*(\alpha, p)$ to be the largest $\rho$ such that the aforementioned upper bound is less than half of the lower bound. According to Theorem 1, $\ell_p$-minimization is now guaranteed to recover all the $\rho^*(\alpha, p)n$-sparse vectors. The problem regarding characterizing the lower bound and the upper bound here is that $B$ has i.i.d. $\mathcal{N}(0, 1)$ entries, and therefore for any $\mathbf{z} \in \mathcal{S}$ and any $T$ and for any constant $c > 0$, there always exist a positive probability that $B\mathbf{z}$ is less than $c$, and similarly a positive probability that $B_T\mathbf{z}$ is greater than $c$. Thus, strictly speaking, no finite value would be a lower bound of $\|B\mathbf{z}\|_p^p$, nor an upper bound of $\|B_T\mathbf{z}\|_p^p$. To address this issue, we will look for a "lower bound" of $\|B\mathbf{z}\|_p^p$ for all $\mathbf{z}$ in $\mathcal{S}$ in Lemma 5 in the sense that the violation probability decays to zero exponentially, and likewise an "upper bound" of $\|B_T\mathbf{z}\|_p^p$ for all $T$ with $|T| = \rho n$ and all $\mathbf{z}$ in $\mathcal{S}$ in Lemma 6 such that the probability it is exceeded decays exponentially to zero. We want the "lower bound (upper bound)" to be as large (small) as possible as long as its violation probability has exponential decay to zero, and we do not focus on the decay rate here. We still define $\rho^*(\alpha, p)$ to be the largest $\rho$ such that the "upper bound" is less than half of the "lower bound". We then show in Theorem 7 that $\ell_p$-minimization can recover all the $\rho^*(\alpha, p)n$-sparse vectors with overwhelming probability.

**Lemma 5.** *For any $\alpha$ and $p$, there exists a constant $\lambda_{\min}(\alpha, p) > 0$ and some constant $c_9 > 0$ such that with probability at least $1 - e^{-c_9 n}$, for every $\mathbf{z} \in \mathcal{S}$, $\|B\mathbf{z}\|_p^p > \lambda_{\min}(\alpha, p)n$.*

**Lemma 6.** *Given any $\alpha$, $p$ and corresponding $\lambda_{\min}(\alpha, p) > 0$, there exists a constant $\rho^*(\alpha, p) > 0$ and some constant $c_{10} > 0$ such that with probability at least $1 - e^{-c_{10} n}$, for every $\mathbf{z} \in \mathcal{S}$ and for every set $T \subset \{1, 2, ..., n\}$ with $|T| \leq \rho^*(\alpha, p)n$, $\|B_T\mathbf{z}\|_p^p < \frac{1}{2}\lambda_{\min}(\alpha, p)n$.*

Together with Lemma 5 and Lemma 6, we are ready to present our result on bounds for strong recovery of $\ell_p$-minimization with given $\alpha \in (0, 1)$.

**Theorem 7.** *For any $0 < p \leq 1$, any $0 < \alpha < 1$, for matrix $A^{m\times n}$ ($\alpha = \frac{m}{n}$) with i.i.d. $\mathcal{N}(0, 1)$ entries, there exists a constant $c_{11} > 0$ such that with probability at least $1 - e^{-c_{11} n}$, $\mathbf{x}$ is the unique solution to the $\ell_p$-minimization problem (2.1) for every vector $\mathbf{x}$ up to $\rho^*(\alpha, p)n$-sparse.*

Theorems 7 implies that for every $\alpha \in (0, 1)$ and every $p \in (0, 1]$, there exists a positive constant $\rho^*(\alpha, p)$ such that $\ell_p$-minimization can recover all the $\rho^* n$-sparse vectors with overwhelming probability. Since $\rho^*(\alpha, p)$ is a lower bound of the threshold of the strong recovery, we would like the lower bound to be as high as possible. Clearly, the value of $\rho^*(\alpha, p)$ depends on the "lower bound" of $\|B\mathbf{z}\|_p^p$ and the "upper bound" of $\|B_T\mathbf{z}\|_p^p$ with $|T| = \rho n$ for a given $\rho$. In order to improve $\rho^*(\alpha, p)$, we need to improve the "lower bound" of $\|B\mathbf{z}\|_p^p$ and the "upper bound" of $\|B_T\mathbf{z}\|_p^p$. Therefore, besides establishing the existence of "lower (upper) bound", we make some efforts to increase (decrease) the "lower (upper) bound" while making sure that the probability of violating these bounds has exponential decay to zero. To be more specific, we first calculate $\lambda_{\min}(\alpha, p)$ in Lemma 5 as a "lower bound" of $\|B\mathbf{z}\|_p^p$. The key idea is as follows. Given any constant $b > 0$, we characterize the probability that $\|B\mathbf{z}\|_p^p \leq bn$ holds for some $\mathbf{z} \in \mathcal{S}$ by techniques like $\gamma$-net arguments, the Chernoff bound and the union bound. Then $\lambda_{\min}(\alpha, p)$ is chosen to be the largest value $b$ such that the probability still maintains exponential decay to zero. With the obtained $\lambda_{\min}(\alpha, p)$, we next calculate $\rho^*(\alpha, p)$ in Lemma 6. The idea is similar to that in calculating $\lambda_{\min}(\alpha, p)$. For any given $\rho > 0$, we calculate an upper bound of the probability that there exists some $\mathbf{z} \in \mathcal{S}$ and some support $T$ with $|T| = \rho n$ such that $\|B_T\mathbf{z}\|_p^p \geq \lambda_{\min}(\alpha, p)n/2$. Then $\rho^*(\alpha, p)$ is chosen to be the largest $\rho$ such that the probability still has exponential decay to zero. Please refer to Appendix-A.10 and Appendix-A.11 for the detailed calculation of $\lambda_{\min}(\alpha, p)$ and $\rho^*(\alpha, p)$.

We numerically compute $\rho^*(\alpha, p)$ by calculating first $\lambda_{\max}(\alpha, p)$ in Lemma 13 from (A.34), and then $\lambda_{\min}(\alpha, p)$ in Lemma 5 from (A.44), and finally $\rho^*(\alpha, p)$ in Lemma 6 from (A.49). Fig. 2.2 shows the curve of $\rho^*(\alpha, p)$ against $\alpha$ for different $p$, and Fig. 2.3 shows the curve of $\rho^*(\alpha, p)$ against $p$ for different $\alpha$. Note that for any $p$, $\lim_{\alpha\to 1} \rho^*(\alpha, p)$ is slightly smaller than the limiting threshold of strong recovery we obtained in Section 2.3.1. For example, when $p = 0.5$, the threshold $\rho^*(0.5)$ we obtained in Section 2.3.1 is 0.3406, and the bound $\rho^*(\alpha, 0.5)$ we obtained here is approximately 0.268 when $\alpha$ goes to 1. This is because in Section 2.3.1 we employed a finer technique to characterize the sum of the largest $\rho n$ terms of $n$ i.i.d. random variables directly, while in Section 2.4.1 introducing the union bound causes some slackness.

Compared with the bound obtained in [11] through restricted isometry condition, our bound $\rho^*(\alpha, p)$ is tighter when $\alpha$ is relatively large. For example, when $p = 0.5$, the bound in [11] (Fig. 4.1 (c)) is in the order of $10^{-3}$ for all $\alpha \in (0, 1)$ and upper bounded by 0.02, while here $\rho^*(\alpha, 0.5)$ is greater than 0.02 for all $\alpha \geq 0.73$ and increases to 0.268 as $\alpha \rightarrow 1$. Therefore, although [11] provides a better bound than ours when $\alpha$ is small, our bound $\rho^*$ improves over that in [11] when $\alpha$ is relatively large.

Chartrand and Staneva [30] provided a lower bound of strong recovery threshold for every $\alpha$ and very $p$. For example, they showed that when $n$ is large enough, $\ell_0$-minimization can recover all the $\frac{\alpha n}{119}$-sparse vectors for given $\alpha$. Their result is better than ours when $\alpha$ is small. However, our bound is higher than that in [30] when $\alpha$ is large. For example, when $\alpha = 0.5$, [30] indicates that a lower bound of recovery threshold in terms of the ratio of sparsity to the dimension $n$ is $0.5/119 \approx 0.004$ for $\ell_0$-minimization. Our result shows that $\rho^*(0.5, 0.7)$ is already 0.004, and $\rho^*(0.5, 0.1)$ is as high as 0.0379, which is approximately ten times the bound 0.5/119 in [30].

Donoho [53] applied geometric face counting technique to the strong bound of successful recovery of $\ell_1$-minimization (Fig. 1.1). Since if the necessary and sufficient condition (2.2) is satisfied for $p = 1$, then it is also satisfied for all $p < 1$, therefore the bound in [53] can serve as the bound of successful recovery for all $0 < p < 1$. Our bound $\rho^*(\alpha, p)$ in Section 2.4 is higher than that in [53] when $\alpha$ is relatively large.

### 2.4.2 Weak Recovery

Theorem 3 provides a sufficient condition for successful recovery of every non-negative $\rho n$-sparse vector $\mathbf{x}$ on one support $T$, which requires $\|B_{T^-}\mathbf{z}\|_p^p < \|B_{T^c}\mathbf{z}\|_p^p$ to hold for all non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$, where given $\mathbf{z}$, $T^- = \{i : B_i\mathbf{z} < 0\}$. We will use arguments similar to those in Section 2.4.1 to obtain a lower bound $\rho_w^*(\alpha, p)$

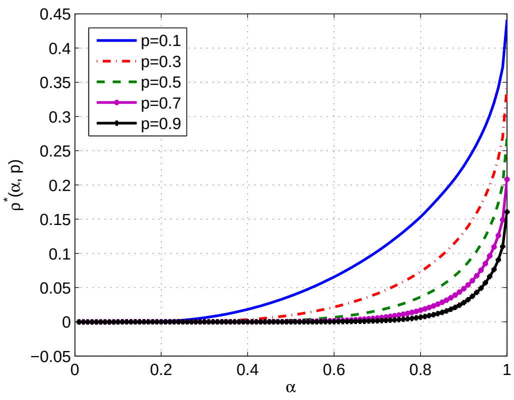



Figure 2.2: $\rho^*(\alpha, p)$ against $\alpha$ for different $p$

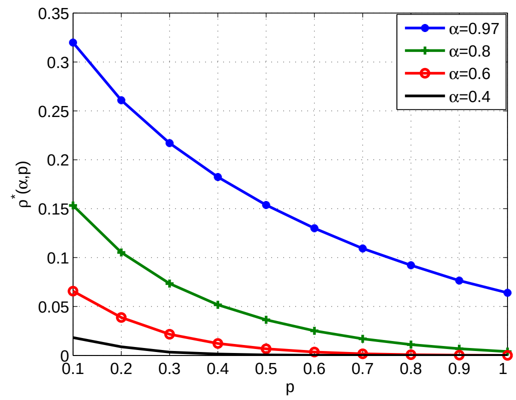



Figure 2.3: $\rho^*(\alpha, p)$ against $p$ for different $\alpha$

of the weak recovery threshold. Given $\alpha$, $p$ and $\rho \in (0, 1)$, we will establish a "lower bound" of $\|B_{T^c}\mathbf{z}\|_p^p$ for all $\mathbf{z} \in \mathcal{S}$ in Lemma 7 in the sense that the violation probability of this "lower bound" decays exponentially to zero, and likewise establish an "upper bound" of $\|B_{T^-}\mathbf{z}\|_p^p$ in Lemma 8. If there exists $\rho_w^*(\alpha, p) > 0$ such that the corresponding "lower bound" of $\|B_{T^c}\mathbf{z}\|_p^p$ is greater than the "upper bound" of $\|B_{T^-}\mathbf{z}\|_p^p$, then $\rho_w^*(\alpha, p)$ serves as a lower bound of recovery threshold of $\ell_p$-minimization for vectors on a fixed support with a fixed sign pattern.

The techniques used to establish the "lower bound" of $\|B_{T^c}\mathbf{z}\|_p^p$ for all $\mathbf{z} \in \mathcal{S}$ is the same as that in Lemma 5. We state the result in Lemma 7, please refer to

Appendix A.13 for its proof.

**Lemma 7.** *Given $\alpha$, $p$ and set $T \subset \{1, ..., n\}$ with $|T| = \rho n$, with probability at least $1 - e^{-c_{12}n}$ for some $c_{12} > 0$, for all $\mathbf{z} \in \mathcal{S}$, $\|B_{T^c}\mathbf{z}\|_p^p < (1-\rho)\lambda_{\max}(\frac{\alpha-\rho}{1-\rho}, p)n$, and with probability at least $1-e^{-c_{13}n}$ for some $c_{13} > 0$, for all $\mathbf{z} \in \mathcal{S}$, $\|B_{T^c}\mathbf{z}\|_p^p > (1-\rho)\lambda_{\min}(\frac{\alpha-\rho}{1-\rho}, p)n$, where $\lambda_{\max}(\alpha, p)$ and $\lambda_{\min}(\alpha, p)$ are defined in (A.34) and (A.44) respectively.*

Given $T$ with $|T| = \rho n$, Lemma 7 provides a "lower bound" of $\|B_{T^c}\mathbf{z}\|_p^p$ which holds with overwhelming probability for all $\mathbf{z} \in \mathcal{S}$. Next we will provide an "upper bound" of $\|B_{T^-}\mathbf{z}\|_p^p$ for all $\mathbf{z} \in \mathcal{S}$ in Lemma 8. One should be cautious that the set $T^-$ varies for different $\mathbf{z}$.

**Lemma 8.** *Given $\alpha$, $p$ and set $T \subset \{1, ..., n\}$ with $|T| = \rho n$, with probability at least $1 - e^{-c_{14}n}$ for some $c_{14} > 0$, for every $\mathbf{z} \in \mathcal{S}$, $\|B_{T^-}\mathbf{z}\|_p^p < \rho\tilde{\lambda}_{\max}(\alpha, p, \rho)n$, for some $\tilde{\lambda}_{\max}(\alpha, p, \rho) > 0$.*

To improve the lower bound of weak recovery threshold, given $\rho$, we want $\tilde{\lambda}_{\max}(\alpha, p, \rho)$ in Lemma 8 to be as small as possible while at the same time the probability that $\|B_{T^-}\mathbf{z}\|_p^p \geq \rho\tilde{\lambda}_{\max}(\alpha, p, \rho)n$ for some $T$ with $|T| = \rho n$ and some $\mathbf{z}$ in $\mathcal{S}$ still has exponential decay to zero. Efforts are made in Appendix A.14 to improve $\tilde{\lambda}_{\max}(\alpha, p, \rho)$, which can be computed from (A.61).

With the help of Lemma 7 and Lemma 8, we are ready to present the result regarding the lower bound of recovery threshold via $\ell_p$-minimization in the weak sense for given $\alpha$.

**Theorem 8.** *For any $0 < p \leq 1$, any $0 < \alpha < 1$, for matrix $A^{m\times n}$ ($m = \alpha n$) with i.i.d. $\mathcal{N}(0,1)$ entries, there exist constants $\rho_w^*(\alpha, p) > 0$ and $c_{15} > 0$ such that with probability at least $1 - e^{-c_{15}n}$, $\mathbf{x}$ is the unique solution to the $\ell_p$-minimization problem (2.1) for every nonnegative $\rho_w^*(\alpha, p)n$-sparse vector $\mathbf{x}$ on fixed support $T$.*

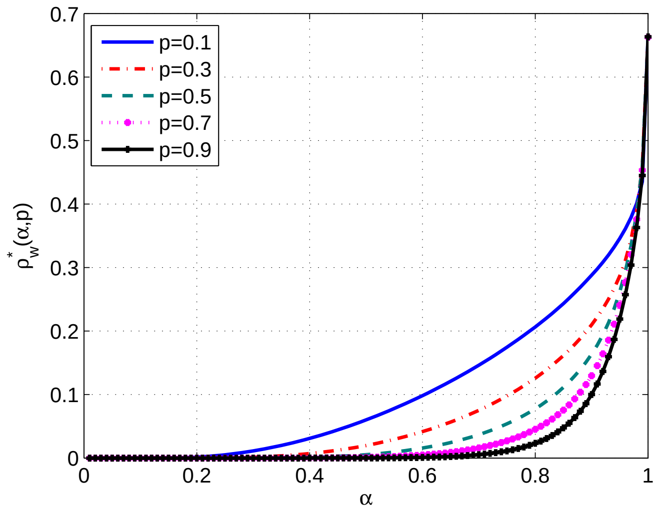



Figure 2.4: $\rho_w^*(\alpha, p)$ against $\alpha$ for different $p$

Theorem 8 establishes the existence of a positive bound $\rho_w^*(\alpha, p)$ of weak recovery threshold. To obtain $\rho_w^*(\alpha, p)$, for every $\rho$ we first calculate $\lambda_{\min}(\frac{\alpha-\rho}{1-\rho}, p)$ in Lemma 7 from (A.44) to obtain a "lower bound" of $\|B_{T^c}\mathbf{z}\|_p^p$ for all $\mathbf{z}$ in $\mathcal{S}$ and calculate $\tilde{\lambda}_{\max}(\alpha, p, \rho)$ in Lemma 8 from (A.61) to obtain an "upper bound" of $\|B_{T^-}\mathbf{z}\|_p^p$ for all $\mathbf{z}$ in $\mathcal{S}$. We then find the largest $\rho_w^*(\alpha, p)$ such that the "lower bound" of $\|B_{T^c}\mathbf{z}\|_p^p$ is larger than the "upper bound" of $\|B_{T^-}\mathbf{z}\|_p^p$, or mathematically, (A.62) holds. We numerically calculate this bound and illustrate the results in Fig. 2.4 and Fig. 2.5. Fig. 2.4 shows the curve of $\rho_w^*(\alpha, p)$ against $\alpha$ for different $p$, and Fig. 2.5 shows the curve of $\rho_w^*(\alpha, p)$ against $p$ for different $\alpha$. When $\alpha \to 1$, $\rho_w^*(\alpha, p)$ goes to $2/3$ for all $p \in (0, 1)$, which coincides with the limiting threshold discussed in Section 2.3.2. As indicated in Fig. 1.2 of [47], the weak recovery threshold of $\ell_1$-minimization is greater than $2/3$ for all $\alpha$ that is greater than 0.9, since the weak recovery threshold of $\ell_p$-minimization ($p \in [0, 1)$) when $\alpha \to 1$ is all $2/3$, therefore for all $\alpha > 0.9$, the weak recovery threshold of $\ell_1$-minimization is greater than that of $\ell_p$-minimization for all $p \in [0, 1)$.

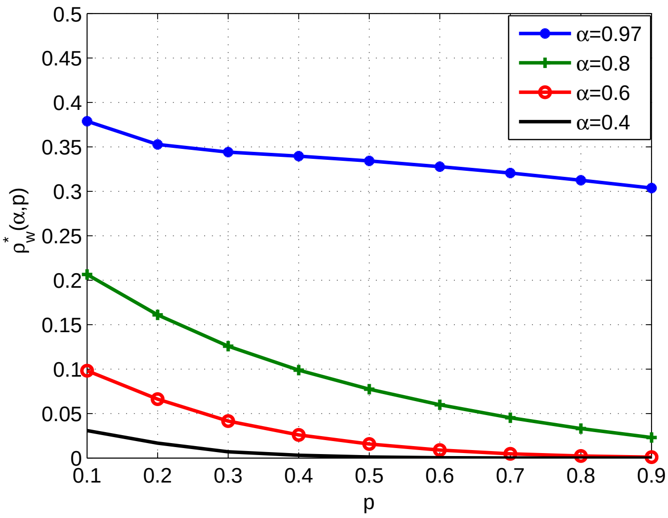



Figure 2.5: $\rho_w^*(\alpha, p)$ against $p$ for different $\alpha$

## 2.5 $\ell_1$-minimization can perform better than $\ell_p$-minimization ($p \in [0, 1)$) for sparse recovery

For strong recovery, if $\ell_1$-minimization can recover all the $k$-sparse vectors, then $\ell_p$-minimization is also guaranteed to recover all the $k$-sparse vectors for all $p \in [0, 1)$. However, for weak recovery, the performance of $\ell_1$-minimization is better than that of $\ell_p$-minimization for all $p \in [0, 1)$ in at least the large $\alpha$ region ($\alpha > 0.9$), and the same result holds for all choices of supports and sign patterns. Then one may naturally ask why $\ell_1$-minimization outperforms $\ell_p$-minimization ($p < 1$) in recovering vectors on every specific support with every specific sign pattern, but is not as good as $\ell_p$-minimization in recovering vectors on all the supports with all the sign patterns? We next provide an intuitive explanation.

Let $\alpha < 1$ be very close to 1, let $n$ be large enough and let $A$ be a random Gaussian matrix. Then with overwhelming probability $\ell_1$-minimization can recover all the vectors up to $\rho_1^s n$-sparse and $\ell_p$-minimization with some $p \in [0, 1)$ can recover all the vectors up to $\rho_p^s n$-sparse, and we know $\rho_1^s < \rho_p^s$ from our discussion on strong bound. Note that since the limiting threshold of strong recovery via $\ell_p$-minimization increases to 0.5 as $p$ decreases to 0, then we have $\rho_1^s < \rho_p^s \leq 0.5$. However, if we only consider the ability to recover all the vectors on one support with one sign pattern, with overwhelming probability $\ell_1$-minimization can recover vectors up to $\rho_1^w n$-sparse, while $\ell_p$-minimization can recover vectors up to $\rho_p^w n$-sparse. From previous discussion about weak recovery threshold, we know that when $\alpha$ is very close to 1, $\rho_1^w > \frac{2}{3} > \rho_p^w > \frac{1}{2}$. And this result holds for any specific choice of the support and the sign pattern. Therefore we have $\rho_1^w > \rho_p^w > \rho_p^s > \rho_1^s$. We illustrate the difference of $\ell_1$ and $\ell_p$-minimization in Fig. 2.6 and Fig. 2.7. Let $\Omega$ be the set of all $m \times n$ matrices with entries drawn from standard Gaussian distribution, and the probability measure $P(\Omega) = 1$. We pick $\rho \in (\rho_1^s, \rho_p^s)$ in Fig. 2.6. Since $\rho < \rho_1^w$, for any fixed support $T_i$ with $|T_i| = \rho n$ and any fixed sign pattern $\sigma_j$, with high probability $\ell_1$-minimization can recover all the $\rho n$-sparse vectors on $T_i$ with sign pattern $\sigma_j$. Let $E_{T_i}^{\sigma_j}$ denote the event that $\ell_1$-minimization can recover all the $\rho n$-sparse vectors on support $T_i$ with sign patter $\sigma_j$. There are $\binom{n}{\rho n}$ different supports, and for each support, there are $2^{\rho n}$ different sign patterns. Then $P(E_{T_i}^{\sigma_j})$ is very close to 1 for every $T_i$ and $\sigma_j$ as shown in Fig. 2.6(a). Since we also have $\rho > \rho_1^s$, then with high probability strong recovery of $\ell_1$-minimization fails, in other words, $\ell_1$-minimization would fail to recover at least one vector with at most $\rho n$ non-zero entries. Let $E$ denote the event that $\ell_1$-minimization can recover all the $\rho n$-sparse vectors, then we have

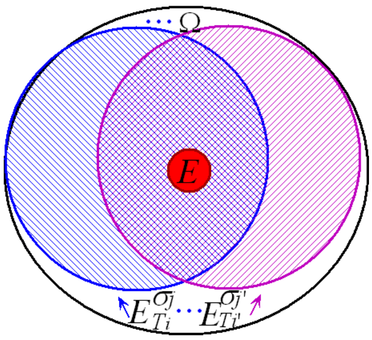
$$E = \bigcap_{i \in \{1, \dots, \binom{n}{\rho n}\}, j \in \{1, \dots, 2^{\rho n}\}} E_{T_i}^{\sigma_j}.$$

Then although $P(E_{T_i}^{\sigma_j})$ is the same for all $T_i$ and $\sigma_j$ and is very close to 1, $P(E)$ is close to 0, as indicated in Fig. 2.6(a). For $\ell_p$-minimization, since $\rho < \rho_p^s$, then with high probability, $\ell_p$-minimization can recover all the $\rho n$-sparse vectors. In Fig.

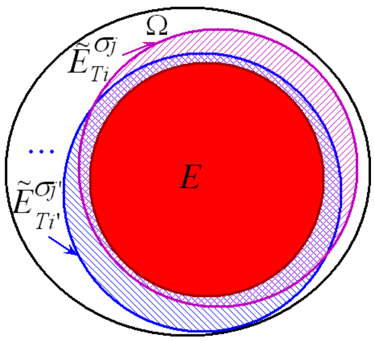
2.6(b), $\tilde{E}$ denotes the event that $\ell_p$-minimization can recover all the $\rho n$-sparse vectors, then

$$\tilde{E} = \bigcap_{i \in \{1,\ldots,\binom{n}{\rho n}\}, j \in \{1,\ldots,2^{\rho n}\}} \tilde{E}_{T_i}^{\sigma_j},$$

where $\tilde{E}_{T_i}^{\sigma_j}$ denotes the event that $\ell_p$-minimization recovers all the vectors on support $T_i$ with sign pattern $\sigma_j$. In this case, $P(\tilde{E})$ is close to 1 as indicated in Fig. 2.6(b). In Fig. 2.7, we pick $\rho \in (\rho_p^w, \rho_1^w)$. Then given any support $T_i$ and any sign pattern $\sigma_j$, $\ell_1$-minimization can recover all the vectors on $T_i$ with sign pattern $\sigma_j$ with high probability, while $\ell_p$-minimization fails to recover at least one vector on $T_i$ with sign pattern $\sigma_j$ with high probability. Therefore $P(E_{T_i}^{\sigma_j})$ is close to 1, while $P(\tilde{E}_{T_i}^{\sigma_j})$ is close to 0 for any given $T_i$ and $\sigma_j$. Therefore, if the sparse vectors we would like to recover are on one same support and share the same sign pattern, $\ell_1$-minimization can be a better choice than $\ell_p$-minimization for all $p \in [0, 1)$ regardless of the amplitudes of the entries of a vector.



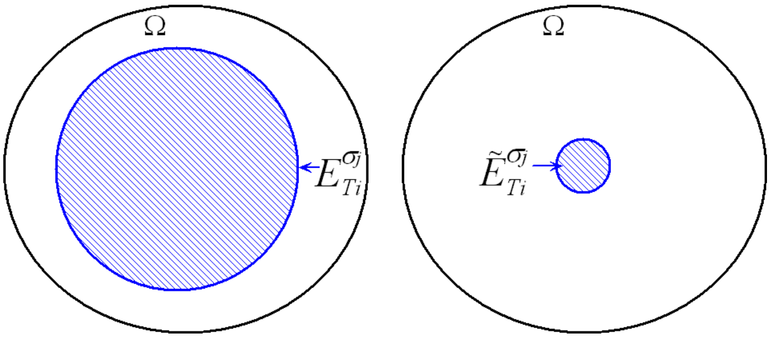
(a) $\ell_1$-minimization



(b) $\ell_p$-minimization

Figure 2.6: Comparison of $\ell_1$ and $\ell_p$-minimization for $\rho \in (\rho_1^s, \rho_p^s)$.

To better understand how the recovery performance changes from strong recovery to weak recovery, let us consider another type of recovery: sectional recovery, which measures the ability of recovering all the vectors on one support $T$. Therefore, the requirement for successful sectional recovery is stricter than that of weak recovery, but is looser than that of strong recovery. The necessary




(a) $\ell_1$-minimization (b) $\ell_p$-minimization

Figure 2.7: Comparison of $\ell_1$ and $\ell_p$-minimization for $\rho \in (\rho_p^w, \rho_1^w)$.

and sufficient condition of successful sectional recovery can be stated as:

**Theorem 9.** *$\ell_p$-minimization problem ($p \in [0,1]$) can recover all the $\rho n$-sparse vectors $\mathbf{x}$ on some support $T$ if and only if*

$$\|B_T\mathbf{z}\|_p^p < \|B_{T^c}\mathbf{z}\|_p^p \tag{2.7}$$

*for all non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$.*

The difference of the null space condition for strong recovery and sectional recovery is that (2.7) should hold for every support $T$ for strong recovery, but only needs to hold for one specific support $T$ for sectional recovery. Though for strong recovery, if the null space condition holds for $p \in [0,1]$, it also holds for all $q \in [0,p]$, this argument is not true for sectional recovery. Consider a simple example that the basis $B$ of null space of $A$ contains only one vector in $\mathcal{R}^4$ and $T = \{1,2\}$. If $B = [16,16,1,36]$, then one can check that $\|B_T\|_1 = 32 < 37 = \|B_{T^c}\|_1$, but $\|B_T\|_{0.5}^{0.5} = 8 > 7 = \|B_{T^c}\|_{0.5}^{0.5}$. If $B = [1,4,1,9]$, then $\|B_T\|_1 < \|B_{T^c}\|_1$, and $\|B_T\|_{0.5}^{0.5} < \|B_{T^c}\|_{0.5}^{0.5}$. Therefore the null space condition of successful sectional recovery holds for $p$ does not necessarily imply that it holds for another $q \neq p$.

Using the techniques as in Section 2.3.2, one can show that when $\alpha \to 1$ and $n$ is large enough, the recovery threshold of sectional recovery is $1/2$ for all $p \in [0, 1]$. We skip the proof here as it follows the lines in Section 2.3.2. To summarize, regarding the recovery threshold when $\alpha \to 1$, $\ell_p$-minimization ($p \in [0, 1]$) has a higher threshold for smaller $p$ for strong recovery; the threshold is $1/2$ for all $p \in [0, 1]$ for sectional recovery; and the threshold is $2/3$ for all $p \in [0, 1)$ and is 1 for $p = 1$ for weak recovery. We can see how recovery performance changes when the requirement for successful recovery changes from strong to weak.

## 2.6 Numerical Experiments

We present the results of numerical experiments to explore the performance of $\ell_p$-minimization. First we consider the special case that the null space of the measurement matrix is only one dimensional. In this case, we can in fact compute the recovery threshold easily.

**Experiment 1. Recovery thresholds when measurement matrices have one-dimensional null space**

The null space of the measurement matrix $A$ is only one-dimensional, and let vector $\beta$ denote the basis of the null space of $A$. Then $\lambda\beta$ is in the null space of $A$ for every $\lambda \in \mathcal{R}$, and every vector in the null space of $A$ can be represented as $\lambda\beta$ for some $\lambda \in \mathcal{R}$. Thus, the strong recovery threshold and the weak recovery threshold of $\ell_1$-minimization and $\ell_p$-minimization can be directly computed by Theorem 1, Theorem 2 and Theorem 3, since we only need to check whether or not the null space condition holds for both $\beta$ and $-\beta$. From Theorem 1, the strong recovery threshold of $\ell_p$-minimization ($p \in (0, 1]$) is the integer $k$ such that the sum of the largest $k$ terms of $|\beta_i|^p$ ($i \in \{1, ..., n\}$) is less than $\|\boldsymbol{\beta}\|_p^p/2$ and the sum of the largest $k+1$ terms of $|\beta_i|^p$ ($i \in \{1, ..., n\}$) is greater than or equal to $\|\boldsymbol{\beta}\|_p^p/2$. For weak recovery, we consider recovering all the nonnegative $k$-sparse vectors on support $T = \{1, ..., k\}$. From Theorem 2, the weak recovery threshold of $\ell_1$-minimization is the largest integer $k$ such that both $\|\boldsymbol{\beta}_{T^-}\|_1 < \|\boldsymbol{\beta}_{T^c}\|_1 + \|\boldsymbol{\beta}_{T^+}\|_1$ and $\|\boldsymbol{\beta}_{T^+}\|_1 < \|\boldsymbol{\beta}_{T^c}\|_1 + \|\boldsymbol{\beta}_{T^-}\|_1$ hold. From Theorem 3, the weak recovery threshold of $\ell_p$-minimization is the largest integer $k$ such that both $\|\boldsymbol{\beta}_{T^-}\|_p^p < \|\boldsymbol{\beta}_{T^c}\|_p^p$ and $\|\boldsymbol{\beta}_{T^+}\|_p^p < \|\boldsymbol{\beta}_{T^c}\|_p^p$ hold.

We generate one hundred random Gaussian matrices $A^{499\times500}$, and for each random matrix $A$, we compute its corresponding strong (and weak) recovery threshold of $\ell_1$ (and $\ell_p$)-minimization. For each $\rho$ between 0 and 1, we count the percentage of random matrices with which $\ell_1$ (and $\ell_p$)-minimization can recover all the $\rho n$-sparse vectors in the strong sense (and in the weak sense). Fig. 2.8 shows the strong recovery thresholds for different $p$ and Fig. 2.9 shows the weak recovery thresholds. We can see that the strong recovery threshold strictly decreases as $p$ increases. However, the weak recovery threshold of $\ell_1$-minimization is close to 0.9, which is greater than the weak recovery threshold of $\ell_p$-minimization for every $p < 1$.

Except for special cases like Experiment 1, (2.1) is indeed non-convex and it is hard to compute its global minimum. In following experiments we employ the iteratively reweighted least squares algorithm [28, 29] to compute the local minimum of (2.1), please refer to [29] about the details of the algorithm.

**Experiment 2. $\ell_p$-minimization using IRLS [29]**

We fix $n = 200$ and $m = 100$, and increase $\rho$ from 0.01 to 0.5. For each $\rho$, we repeat the following procedure one hundred times. We first generate an

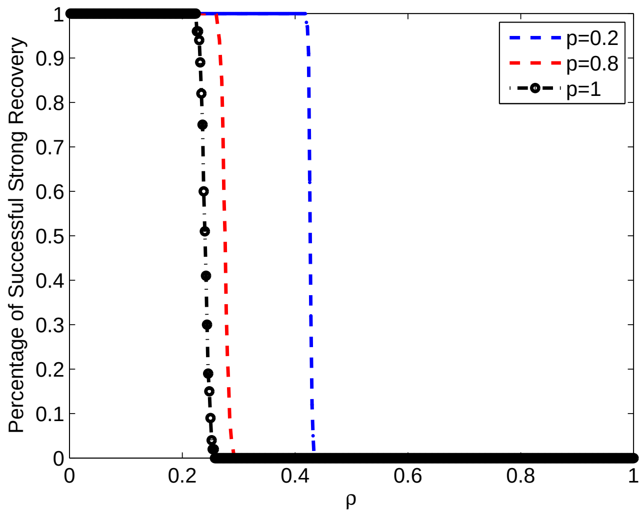



Figure 2.8: Strong recovery threshold with 499 × 500 Gaussian matrix

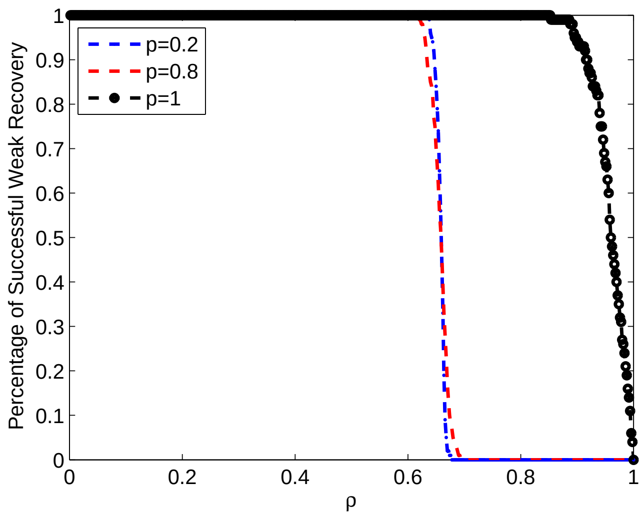



Figure 2.9: Weak recovery threshold with 499 × 500 Gaussian matrix

$n$-dimensional vector $\mathbf{x}$ with $\rho n$ non-zero entries. The location of the non-zero entries are chosen randomly, and each non-zero value follows from standard Gaussian distribution. We then generate an $m \times n$ matrix $A$ with i.i.d. $\mathcal{N}(0, 1)$ entries. We let $\mathbf{y} = A\mathbf{x}$ and run the iteratively reweighted least squares algorithm to search for a local minimum of (2.1) with $p$ chosen to be 0.2, 0.5, and 0.8 respectively. Let $\mathbf{x}^*$ be the output of the algorithm, if $\|\mathbf{x}^* - \mathbf{x}\|_2 \leq 10^{-4}$, we

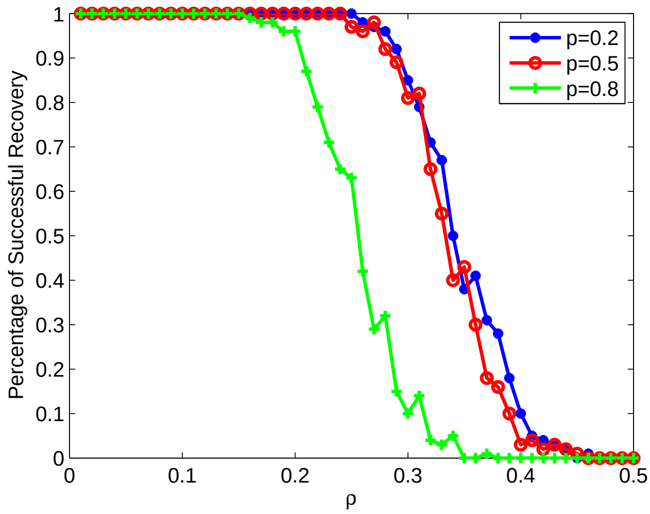



Figure 2.10: Successful recovery of $\rho n$-sparse vectors via $\ell_p$-minimization

say the recovery of $\mathbf{x}$ is the successful. Fig. 2.10 records the percentage of times that the recovery is successful for different sparsity $\rho n$. Note that the iteratively reweighted least squares algorithm is designed to obtain a local minimum of the $\ell_p$-minimization problem (2.1), and is not guaranteed to obtain the global minimum. However, as shown in Figure 2.10, it indeed recovers the sparse vectors up to certain sparsity. For $\ell_{0.2}$, $\ell_{0.5}$ and $\ell_{0.8}$-minimization computed by the heuristic, the sparsity ratios of successful recovery are 0.25, 0.24, and 0.15 respectively.

**Experiment 3. Strong recovery vs. weak recovery**

We also compare the performance of $\ell_p$-minimization and $\ell_1$-minimization both for strong recovery in Fig. 2.11 and for weak recovery in Fig. 2.12 when $\alpha$ is large. We employ the trial version of `MOSEK` [98] to solve $\ell_1$-minimization and still employ the iteratively reweighted least squares algorithm to compute a local minimum of $\ell_p$-minimization. We fix $n = 60$ and $m = 58$ and independently generate one hundred random matrices $A^{m\times n}$ with i.i.d. $\mathcal{N}(0,1)$ entries and eval-

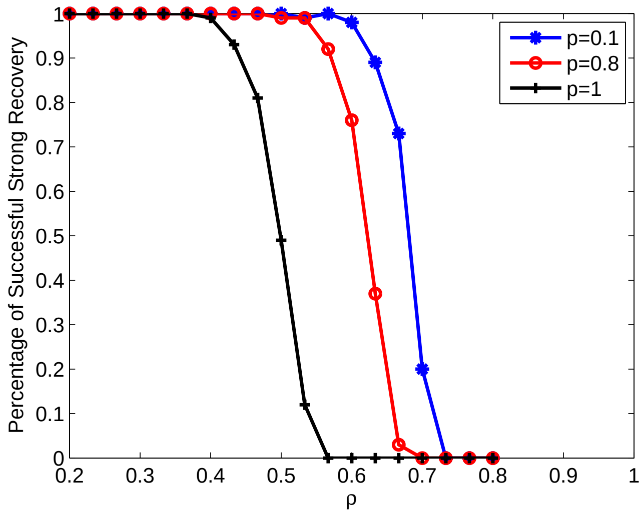



Figure 2.11: Successful strong recovery of $\rho n$-sparse vectors

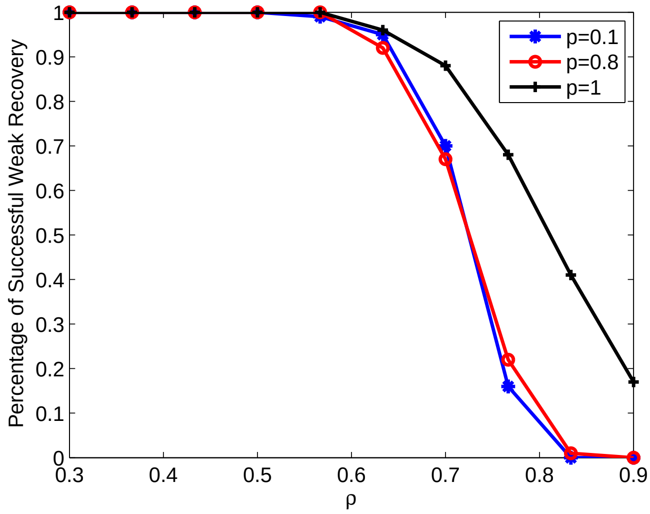



Figure 2.12: Successful weak recovery of $\rho n$-sparse vectors

uate the performance of strong recovery and weak recovery. For each matrix, we increase $\rho$ from 0.2 to 1. In weak recovery, we consider recovering nonnegative vectors on a fixed support $T = \{1, ..., \rho n\}$. For a given $\rho$, we generate one thousand vectors and claim the weak recovery of $\rho n$-sparse vectors to be successful if and only if all the vectors are successfully recovered. For each vector $\mathbf{x}$, $x_i = |z_i|$ $(i \in T)$, and $z_i$ is generated from $\mathcal{N}(0, 1)$ with probability 0.5, and $\mathcal{N}(1000, 1)$ with probability 0.5. As discussed in Section 2.2, the condition for successful weak recovery via $\ell_1$-minimization is the same for every nonnegative vector on $T$, therefore for a fixed matrix $A$, if $\ell_1$-minimization recovers all the vectors we generated, it should also recover all the nonnegative vectors on $T$. $\ell_p$-minimization ($p \in [0, 1)$), on the other hand, can recover some nonnegative vectors on $T$ while at the same time fails to recover some other nonnegative vectors on $T$. Therefore, since we could not check every nonnegative $\mathbf{x}$ on $T$, $\ell_p$-minimization ($p < 1$) can still fail to recover some other nonnegative vector on $T$ even if we declare the weak recovery to be "successful". In strong recovery, for each $\rho$, we generate one thousand vectors and claim the strong recovery to be successful if and only if all these vectors are correctly recovered. For each such random $\rho n$-sparse vector $\mathbf{x}$, we first randomly pick a support $T$ with $|T| = \rho n$, and then for each $x_i$ ($i \in T$), $x_i$ is generated from $\mathcal{N}(0, 1)$ with probability 0.5, from $\mathcal{N}(1000, 1)$ with probability 0.25, and from $\mathcal{N}(-1000, 1)$ with probability 0.25. The average performance of one hundred random matrices for strong recovery is plotted in Fig. 2.11, and the average performance of weak recovery is plotted in Fig. 2.12. Note that we only apply iteratively reweighted least squares algorithm to approximate the performance of $\ell_p$-minimization, therefore the solution returned by the algorithm may not always be the solution of $\ell_p$-minimization. Simulation results indicate that for strong recovery, the recovery threshold increases as $p$ decreases, while for the weak recovery, interestingly, the recovery threshold of $\ell_1$-minimization is higher than any other $\ell_p$-minimization for $p < 1$.

CHAPTER 3

# SPARSE RECOVERY WITH GRAPH CONSTRAINTS

In the literature of sparse recovery, there is no constraint on the measurement matrix, and in fact any real-valued matrix could be a potential measurement matrix. Random Gaussian matrices and random Bernoulli matrices are known to be good measurement matrices for sparse recovery. In Chapter 2, we discussed the fundamental limits of sparse recovery via $\ell_p$-minimization for $p \in [0, 1)$, and our analysis are tailored to random gaussian matrices.

Motivated by the need to monitor large-scale networks, this chapter addresses the problem of sparse recovery with additional topological constraints. Unlike conventional sparse recovery where a measurement can contain any subset of the unknown variables(nodes), we take an additive measurement over nodes only if they satisfy certain topological constraints. In applications such as network monitoring, one may take measurements over objects only if they form a path or a cycle in the network. Given topological constraints, we construct measurements satisfying such constraints such that one can still recover sparse signals from a small number of measurements.

## 3.1 Introduction

Network monitoring is a critical module in the operation and management of communication networks, where one keeps track of network state parameters, such as bandwidth utilization and queueing delay. Since measuring each component (e.g., router) in the network directly can be operationally costly, if feasible at all, the topic of inferring system internal characteristics from indirect end-to-end (aggregate) measurements becomes important. This area is known as network tomography, and has been extensively studied during the last decade or so [17, 32, 35, 61, 72, 102, 146].

In many cases, the total number of aggregate measurements is much smaller than the number of components in a network. But we still hope to extract the status of each individual component with some prior knowledge of the unknown signal to recover. For instance, if the signal is sparse, i.e. most entries are zero, we can recover it exactly even though the number of measurements is much smaller than its dimension. For example, transmission delays in the communication networks can be represented by an approximately sparse signal, since only a small number of bottleneck links experience large delays, while the delay is approximately zero elsewhere. That connects network monitoring with sparse recovery. Sparse Recovery addresses the problem of recovering sparse signals from a smaller number of measurements, and has two different but closely related problem formulations. One is Compressed Sensing [8, 20, 21, 47, 55, 66], where the signal is represented by a high-dimensional real vector, and an aggregate measurement is the arithmetical sum of the corresponding real entries. For example, the unknown sparse vector represents the transmission delays at all links, and a path delay measurement records the sum of delays on links it passes. The other is Group Testing [57, 58], where the high-dimensional signal is binary and a measurement is a logical disjunction (**OR**) on the corresponding binary values. For example, in all-optical networks, the success and the failure of an link is represented by '0' and '1' respectively. If a measurement does not pass through any failed links, we claim it be to be successful, denoted by '0'. If it passes at least one failed link, we claim it be a failure, denoted by '1'.

One key question in both compressed sensing and group testing is to design a small number of non-adaptive measurements (either real or logical) such that all the vectors (either real or logical) up to certain sparsity (the support size of a vector) can be correctly recovered. Most existing results, however, rely critically on the assumption that any subset of the values can be aggregated together [20, 47], which is not realistic in network monitoring problems where only objects that form a path or a cycle on the graph [1, 72], or induce a connected subgraph can be aggregated together in the same measurement. Only a few recent works consider graph topological constraints, either in group testing [33] setup, especially motivated by link failure localization in all-optimal networks [4, 33, 75, 121, 136], or in compressed sensing setup, with application in estimation of network parameters [36, 66, 76, 142].

Though motivated by network monitoring problems, sparse recovery with graph constraints abstractly models scenarios when certain elements cannot be measured together in a complex system. These constraints can result from various reasons, not necessarily lack of connectivity. Therefore, our results can be potentially useful to other applications besides network tomography.

Here are the main contributions of this chapter.

**(1)** We provide explicit measurement constructions for various graphs. Our construction for line networks is optimal in the sense that it requires the minimum number of measurements. For other special graphs, the number measurements by our construction is less than the existing estimates (e.g. [33, 142]) of the number of measurements required to recover sparse vectors over graphs. (Section

3.3)

**(2)** We propose a measurement design guideline based on *r-partition* for general graphs and further show some of its properties. (Section 3.4.1)

**(3)** A simple measurement design algorithm is proposed for general graphs, and we evaluate its performance both theoretically and numerically. (Section 3.4.2 and 3.8)

**(4)** For Erdős-Rényi random graphs, we characterize the dependence of the number of measurements for sparse recovery on the graph structure. (Section 3.5)

**(5)** Motivated by practical needs, we further propose measurement construction methods under additional graph constraints including measurement length constraint, and the requirement that each measurement should pass one of a fixed set of nodes. (Section 3.6) We also address the issue of sparse recovery when some critical measurements may contain errors. (Section 3.7)

## 3.2 Model and Problem Formulation

Consider a graph $G = (V, E)$, where $V$ denotes the set of nodes with cardinality $|V| = n$ and $E$ denotes the set of links. Each node $i$ is associated with a real number $x_i$, and we say vector $\mathbf{x} = (x_i, i = 1, ..., n)$ is associated with $G$.

Let $S \subseteq V$ denote a subset of nodes in $G$. Let $E_S$ denote the subset of links with both ends in $S$, then $G_S = (S, E_S)$ is the induced subgraph of $G$. We have the following two assumptions throughout this chapter:

**(A1)**: A set $S$ of nodes can be measured together in one measurement if and only if $G_S$ is connected.

**(A2)**: The measurement is an additive sum of values at the corresponding nodes.

(A1) captures the graph constraints. One practical example is a sensor network where the nodes represent sensors and the links represent feasible communication between sensors. For a set $S$ of nodes that induce a connected subgraph, one node $u$ in $S$, we call $u$ an "agent", monitors the sum of node values in $S$. Every node in $S$ obtains values from its children, if any, on the spanning tree rooted at $u$, sums them up with its own value and sends the sum to its parent. Then the central operator can obtain the sum of node values in $S$ by only communicating with the agent $u$. (A2) follows from the additive property of many network characteristics, e.g. delays and packet loss rates [72]. Note that compressed sensing can also be applied to cases where (A2) does not hold, e.g., the measurements can be nonlinear as in [12, 127].

Let $\mathbf{y} \in \mathcal{R}^m$ ($m \ll n$) denote the vector of $m$ measurements. $A$ is an $m \times n$ measurement matrix with its $i$th row corresponding to the $i$th measurement, i.e., $A_{ij} = 1$ ($i = 1, ..., m,\ j = 1, ..., n$) if and only if node $j$ is included in the $i$th measurement and $A_{ij} = 0$ otherwise. We can write in the compact form that $\mathbf{y} = A\mathbf{x}$. We say a measurement matrix $A$ can ***identify all k-sparse vectors*** if and only if $A\mathbf{x}_1 \neq A\mathbf{x}_2$ for every two different vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ that are at most $k$-sparse. This definition indicates that every vector $\mathbf{x}$ that is at most $k$-sparse can be recovered from $A\mathbf{x}$ via $\ell_0$-minimization, which returns the sparsest vector among all vectors that can produce the same observation $A\mathbf{x}$. Sparse recovery theory indicates that it is possible to identify $n$-dimensional vectors from $m$ ($m \ll n$) measurements provided that the vectors are sparse enough.

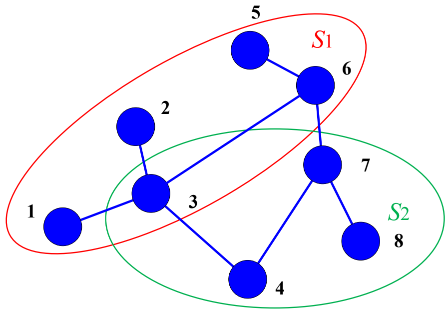



Figure 3.1: Network Example

In conventional compressed sensing, any real-valued matrix can be a measurement matrix. Here, with the above assumptions on graph constraints, $A$ is a 0-1 matrix, and for each row of $A$, the set of nodes that correspond to '1' should form a connected induced subgraph of $G$. For example in Fig. 3.1, we can measure the sum of nodes in $S_1$ and $S_2$ by two separate measurements, and the measurement matrix is

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

We remark here that in group testing with graph constraints, the requirements for the measurement matrix $A$ are the same, while group testing differs from compressed sensing only in that (1) $\mathbf{x}$ is a logical vector, and (2) the operations used in each group testing measurement are the logical "AND" and "OR". Here we consider compressed sensing if not otherwise specified, and the main results are stated in theorems. We sometimes discuss group testing for comparison, and the results are stated in propositions. Note that for recovering 1-sparse vectors, the numbers of measurements required by compressed sensing and group testing are the same.

Given a graph $G$ with $n$ nodes, let $M^G_{k,n}$ denote the minimum number of non-adaptive measurements needed to identify all $k$-sparse vectors associated with

Table 3.1: summary of key notations

| Notation | Meaning |
|---|---|
| $G_S$ | Subgraph of $G$ induced by $S$ |
| $M^G_{k,n}$ | Minimum number of measurements needed to identify $k$-sparse vectors associated with $G$ of $n$ nodes. |
| $M^C_{k,n}$ | Minimum number of measurements needed to identify $k$-sparse vectors associated with a complete graph of $n$ nodes. |
| $f(k,n)$ | Number of measurements constructed to identify $k$-sparse vectors associated with a complete graph of $n$ nodes |

$G$. Let $M^C_{k,n}$ denote the minimum number of non-adaptive measurements needed in a complete graph with $n$ nodes. Since in a complete graph, any subset of nodes forms a connected subgraph, every 0-1 matrix is a feasible measurement matrix there. Existing results [8, 21, 138] show that with overwhelming probability a random 0-1 matrix with $O(k\log(n/k))$ rows[1] can identify all $k$-sparse vectors associated with a complete graph, and we can recover the sparse vector by $\ell_1$-minimization, which returns the vector with the least $\ell_1$-norm[2] among those that can produce the obtained measurements. Then we have

$$M^C_{k,n} = O(k\log(n/k)). \tag{3.1}$$

Explicit constructions of measurement matrices for complete graphs also exist, e.g., [3, 8, 40, 46, 138]. We use $f(k,n)$ to denote the number of measurements to recover $k$-sparse vectors associated with a complete graph of $n$ nodes by a particular measurement construction method. $f(k,n)$ varies for different construction methods, and clearly $f(k,n) \geq M^C_{k,n}$. Table 3.1 summarizes the key notations.

[1]We use the notations $g(n) \in O(h(n))$, $g(n) \in \Omega(h(n))$, or $g(n) = \Theta(h(n))$ if as $n$ goes to infinity, $g(n) \leq ch(n)$, $g(n) \geq ch(n)$ or $c_1h(n) \leq g(n) \leq c_2h(n)$ eventually holds for some positive constants $c$, $c_1$ and $c_2$ respectively.

[2]The $\ell_p$-norm ($p \geq 1$) of $\mathbf{x}$ is $\|\mathbf{x}\|_p = (\sum_i |x_i|^p)^{1/p}$, $\|\mathbf{x}\|_\infty = \max_i |x_i|$, and $\|\mathbf{x}\|_0 = |\{i : x_i \neq 0\}|$.

The questions we would like to address in the chapter are:

- Given graph $G$, what is the corresponding $M^G_{k,n}$? What is the dependence of $M^G_{k,n}$ on $G$?
- How can we explicitly design measurements such that the total number of measurements is close to $M^G_{k,n}$?

## 3.3 Sparse Recovery over Special Graphs

In this section, we consider four kinds of special graphs: one-dimensional line/ring network, ring with each node connecting to its four closest neighbors, two-dimensional grid and a tree. The measurement construction method for a line/ring network is different from those for the other graphs, and our construction is optimal (or near optimal) for a line (or ring) network. For other special graphs, we construct measurements based on the "hub" idea and will later extend it to general graphs in Section 3.4.

### 3.3.1 Line and Ring

First consider a line/ring network as shown in Fig. 3.2. When later comparing the results here with those in Section 3.3.2, one can see that the number of measurements required for sparse recovery can be significantly different in two graphs that only differ from each other with a small number of links.

In a line/ring network, there is not much freedom in the measurement design since only consecutive nodes can be measured together from (A1). Recov-

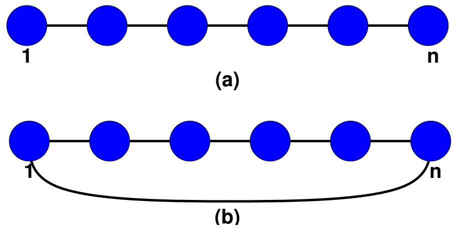



Figure 3.2: (a) line network (b) ring network

ering 1-sparse vectors associated with a line (or ring) network with $n$ nodes is considered in [75, 121], which show that $\lceil \frac{n+1}{2} \rceil$ (or $\lceil \frac{n}{2} \rceil$) measurements are both necessary and sufficient in this case. Here, we consider recovering $k$-sparse vectors for $k \geq 2$.

Our construction works as follows. Given $k$ and $n$, let $t = \lfloor \frac{n+1}{k+1} \rfloor$. We construct $n+1-\lfloor \frac{n+1}{k+1} \rfloor$ measurements with the $i$th measurement passing all the nodes from $i$ to $i+t-1$. Let $A^{(n+1-t)\times n}$ be the corresponding measurement matrix, then its $i$th row has '1's from entry $i$ to entry $i+t-1$ and '0's elsewhere. For example, when $k=3$ and $n=11$, we have $t=3$, and

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}. \tag{3.2}$$

Let $M^L_{k,n}$ and $M^R_{k,n}$ denote the minimum number of measurements required to recover $k$-sparse vectors in a line/ring network respectively. We have

**Theorem 10.** *Our constructed $n+1-\lfloor\frac{n+1}{k+1}\rfloor$ measurements can identify all k-sparse vectors associated with a line/ring network with* n *nodes. Moreover,*

$$M^L_{k,n} = n+1-\lfloor\frac{n+1}{k+1}\rfloor \leq M^R_{k,n}+1. \tag{3.3}$$

*Proof. We first prove that the constructed* $n+1-\lfloor\frac{n+1}{k+1}\rfloor$ *measurements can identify all k-sparse vectors associated with a line/ring network with* n *nodes.*

The recovery is successful if and only if every $2k$ columns of $A$ are linearly independent, i.e., every non-zero vector $\mathbf{z}$ such that $A\mathbf{z} = \mathbf{0}$ has at least $2k+1$ non-zero elements [20].

$A$ is an identity matrix when $t = 1$, and the argument holds trivially. We focus on the case that $t \geq 2$. Now $tk \leq n-1$ holds from definition. For each index $1 \leq k' \leq k-1$, define a submatrix $A_{k'}$, which consists of the first $tk'+1$ rows and the first $tk'+t$ columns of $A$. For example, for $A$ in (3.2),

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

We claim that: (*) every non-zero vector $\mathbf{w}$ such that $A_{k'}\mathbf{w} = \mathbf{0}$ has at least $2k' + 2$ non-zero elements, and at least two non-zero elements exist in its last $t$ entries. We will prove (*) by induction over $k'$.

First consider $A_1$. Let $\mathbf{a}_i^T$ denote its $i$th row. Because any two columns of $A_1$ are linearly independent, any $\mathbf{w} \neq \mathbf{0}$ such that $A_1\mathbf{w} = \mathbf{0}$ must have at least three non-zero elements. Let $j$ be the index of the last non-zero element of $\mathbf{w}$. Suppose $j \leq t$, then $\mathbf{a}_j^T\mathbf{w} = w_j \neq 0$, contradicting the fact that $A_1\mathbf{w} = \mathbf{0}$. Then $j \geq t + 1$ must hold, i.e., at least one of the last $t$ entries of $\mathbf{w}$ is non-zero. From $\mathbf{a}_{t+1}^T\mathbf{w} = 0$, at least two non-zero elements exist in the last $t$ entries of $\mathbf{w}$. One can similarly argue that at least two non-zero elements exist in the first $t$ entries of $\mathbf{w}$. Thus, (*) holds for $A_1$.

Now suppose (*) holds for $A_{k'}$ with integer $k'$ in $[1, k-2]$. Consider $\mathbf{w} \neq \mathbf{0}$ such that $A_{k'+1}\mathbf{w} = \mathbf{0}$. Let $\hat{\mathbf{w}}$ denote the subvector of the first $tk' + t$ entries of $\mathbf{w}$, then $A_{k'}\hat{\mathbf{w}} = \mathbf{0}$. We remark that $\hat{\mathbf{w}} \neq \mathbf{0}$. Suppose $\hat{\mathbf{w}} = \mathbf{0}$, and let $j$ denote the index of the first non-zero entry of $\mathbf{w}$. Then the inner prodcuct of the $(j+1-t)$th row of $A_{k'+1}$ with $\mathbf{w}$ equals to $w_j$, which is non-zero, contracting the fact that $A_{k'+1}\mathbf{w} = \mathbf{0}$.

Since $\hat{\mathbf{w}} \neq \mathbf{0}$, from the induction hypothesis, it has at least $2k' + 2$ non-zero elements, and at least two non-zero elements in its last $t$ entries. Now consider the last $2t$ entries of $\mathbf{w}$ and the last $t + 1$ rows of $A_{k'+1}$. From a similar argument as for $A_1$, we know that $\mathbf{w}$ must have at least two non-zero elements in the last $t$ entries. So $\mathbf{w}$ has at least $2(k' + 1) + 2$ non-zero elements, and (*) holds for $k' + 1$.

By induction over $k'$, we conclude that (*) follows. Consider any $\mathbf{z} \neq \mathbf{0}$ such that $A\mathbf{z} = \mathbf{0}$. Let $\mathbf{w}$ contain its first $kt$ entries. One can argue that $\mathbf{w} \neq \mathbf{0}$. Then by (*), $\mathbf{w}$ has at least $2k$ non-zero entries. Let $j$ denote the index of the last non-zero entry of $\mathbf{w}$. Since the inner product between $\mathbf{z}$ and the $j$th row of $A$ is zero, there exists at least one non-zero entry in the last $n-kt$ entries of $\mathbf{z}$. Thus, $\mathbf{z}$ has at least $2k+1$ non-zero entries in total. This concludes the proof.

*We next prove that the number of measurements needed to recover k-sparse vectors associated with a line (or ring) networks is at least* $n+1-\lfloor\frac{n+1}{k+1}\rfloor$ *(or* $n-\lfloor\frac{n}{k+1}\rfloor$*.)*

Let $A^{m\times n}$ denote a measurement matrix with which one can recover $k$-sparse vectors associated with a line network with $n$ nodes. Then every $2k$ columns of $A$ must be linearly independent. We will prove that $m \geq n+1-\lfloor\frac{n+1}{k+1}\rfloor$.

Let $\boldsymbol{\beta}^i$ denote the $i$th column of $A$. Define $\boldsymbol{\alpha}^1 = \boldsymbol{\beta}^1$, $\boldsymbol{\alpha}^i = \boldsymbol{\beta}^i - \boldsymbol{\beta}^{i-1}$ for all $2 \leq i \leq n$, and $\boldsymbol{\alpha}^{n+1} = -\boldsymbol{\beta}^n$. Define matrix $P^{m\times(n+1)} = (\boldsymbol{\alpha}^i, 1 \leq i \leq n+1)$. Since $A$ is a measurement matrix for a line network, each row of $P$ contains one '1' entry and one ' − 1' entry, and all the other entries must be '0's.

Given $P$, we construct a graph $G_{eq}$ with $n+1$ nodes as follows. For every row $i$ of $P$, there is an edge $(j,k)$ in $G_{eq}$, where $P_{ij} = 1$ and $P_{ik} = -1$. Then $G_{eq}$ contains $m$ edges, and $P$ can be viewed as the transpose of an oriented incidence matrix of $G_{eq}$. Let $S$ denote the set of indices of nodes in a component of $G_{eq}$, then one can check that

$$\sum_{i\in S} \boldsymbol{\beta}^i = \mathbf{0}. \tag{3.4}$$

Since every $2k$ columns of $A$ are linearly independent, every $k$ columns of $P$ are linearly independent, which then implies that the sum of any $k$ columns of $P$ is not a zero vector. With (3.4), we know that any component of $G_{eq}$ should have at least $k+1$ nodes. Since a component with $r$ nodes contains at least $r-1$ edges, and $G_{eq}$ has at most $\lfloor\frac{n+1}{k+1}\rfloor$ components, then $G_{eq}$ contains at least $n+1-\lfloor\frac{n+1}{k+1}\rfloor$ edges. The claim follows.

We next consider the ring network. Let $\tilde{A}$ denote the measurement matrix with which one can recover $k$-sparse vectors on a ring network with $n$ nodes. Let $\tilde{\beta}^i$ denote the $i$th column of $\tilde{A}$. Define $\tilde{\alpha}^1 = \tilde{\beta}^1 - \tilde{\beta}^n$, and $\tilde{\alpha}^i = \tilde{\beta}^i - \tilde{\beta}^{i-1}$ for all $2 \leq i \leq n$. Define matrix $\tilde{P}^{m\times n} = (\tilde{\alpha}^i, 1 \leq i \leq n)$. Similarly, we construct a graph $\tilde{G}_{eq}$ with $n$ nodes based on $\tilde{P}$, and each component of $\tilde{G}_{eq}$ should have at least $k+1$ nodes. Thus, $\tilde{G}_{eq}$ contains at most $\lfloor \frac{n}{k+1} \rfloor$ components, and has at least $n - \lfloor \frac{n}{k+1} \rfloor$ edges. Then we have

$$M^R_{k,n} \geq n - \lfloor \frac{n}{k+1} \rfloor \geq n - \lfloor \frac{n+1}{k+1} \rfloor,$$

and the inequality of (3.3) holds. □

Theorem 10 indicates that our construction is optimal for a line network in the sense that the number of measurements is equal to the minimum needed to recover $k$-sparse vectors. It is near optimal for a ring network, since the number of measurements is no more than the minimum plus one. This improves over our earlier result (Theorem 1 in [130]), which does not have optimality guarantee.

We can save about $\lfloor \frac{n+1}{k+1} \rfloor - 1$ measurements but still be able to recover $k$-sparse vectors in a line/ring network via compressed sensing. But for group testing on a line/ring network, $n$ measurements are necessary to recover more than one non-zero element. The key is that every node should be the *endpoint* at least twice, where the endpoints are the nodes at the beginning and the end of a measurement. If node $u$ is an endpoint for at most once, then it is always measured together with one of its neighbors, say $v$, if ever measured. Then when $v$ is '1', we cannot determine the value of $u$, either '1' or '0'. Therefore, to recover more than one non-zero element, we need at least $2n$ endpoints, and thus $n$ measurements.

### 3.3.2 Ring with nodes connecting to four closest neighbors

Consider a graph with each node directly connecting to its four closest neighbors as in Fig. 3.4 (a), denoted by $\mathcal{G}^4$. $\mathcal{G}^4$ is important to the study of small-world networks [132]. $\mathcal{G}^4$ has $n$ more links than the ring network, but we will show that the number of measurements required by compressed sensing to recover $k$-sparse vectors associated with $\mathcal{G}^4$ significantly reduces from $\Theta(n)$ to $O(k \log(n/k))$. The main idea is referred to as " the use of a hub".

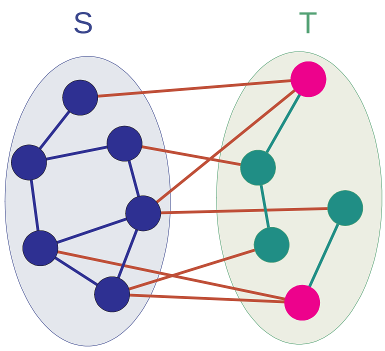
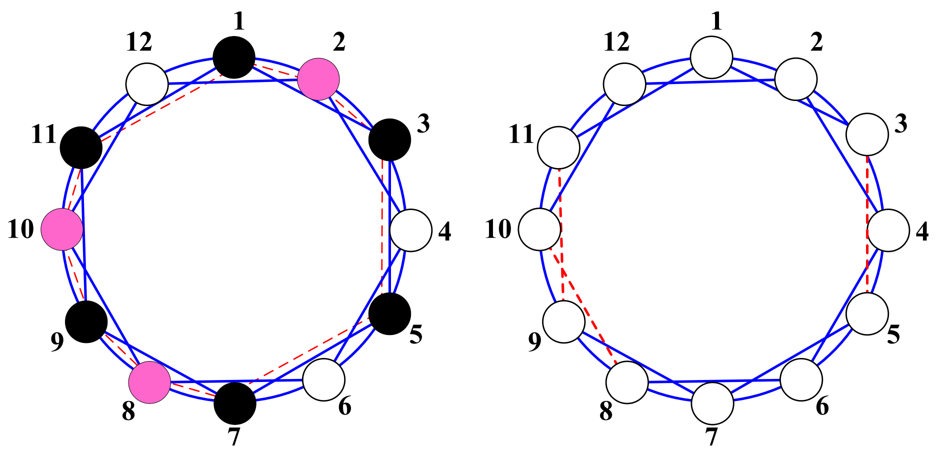



Figure 3.3: Hub $S$ for $T$

**Definition 2.** *Given $G = (V, E)$, $S \subseteq V$ is a **hub** for $T \subseteq V$ if $G_S$ is connected, and $\forall u \in T, \exists s \in S$ s.t. $(u, s) \in E$.*

We first take one measurement of the sum of nodes in $S$, denoted by $s$. For any subset $W$ of $T$, e.g., the pink nodes in Fig. 3.3, $S \cup W$ induces a connected subgraph from the hub definition and thus can be measured by one measurement. To measure the sum of nodes in $W$, we first measure nodes in $S \cup W$ and then subtract $s$ from the sum. Therefore we can apply the measurement constructions for complete graphs on $T$ with this simple modification, and that requires only one additional measurement for the hub $S$. Thus,




(a) Measure nodes 2,8 and 10 via hub $T_o$

(b) Delete $h$ long links

Figure 3.4: Sparse recovery on graph $\mathcal{G}^4$

**Theorem 11.** *With hub $S$, $M^C_{k,|T|} + 1$ measurements are enough to recover k-sparse vectors associated with $T$.*

The significance of Theorem 11 is that $G_T$ is not necessarily a complete subgraph, i.e., a clique, and it can even be disconnected. As long as there exists a hub $S$, the measurement construction for a complete graph with the same number of nodes can be applied to $T$ with simple modification. Our later results rely heavily on Theorem 11.

In $\mathcal{G}^4$, if nodes are numbered consecutively around the ring, then the set of all the odd nodes, denoted by $T_o$, form a hub for the set of all the even nodes, denoted by $T_e$. Given a $k$-sparse vector $\mathbf{x}$, let $\mathbf{x}_o$ and $\mathbf{x}_e$ denote the subvectors of $\mathbf{x}$ with odd and even indices. Then $\mathbf{x}_o$ and $\mathbf{x}_e$ are both at most $k$-sparse. From Theorem 11, $M^C_{k,\lfloor n/2 \rfloor}+1$ measurements are enough to recover $\mathbf{x}_e \in \mathcal{R}^{\lfloor n/2 \rfloor}$. Similarly, we can use $T_e$ as a hub to recover the subvector $\mathbf{x}_o \in \mathcal{R}^{\lceil n/2 \rceil}$ with $M^C_{k,\lceil n/2 \rceil} + 1$ measurements, and thus $\mathbf{x}$ is recovered.

**Corollary 3.** *All k-sparse vectors associated with $\mathcal{G}^4$ can be recovered with $M^C_{k,\lfloor n/2 \rfloor} + M^C_{k,\lceil n/2 \rceil} + 2$ measurements, which is $O(2k \log(n/(2k)))$.*

From ring network to $\mathcal{G}^4$, although the number of links only increases by $n$, the number of measurements required to recover $k$-sparse vectors significantly reduces from $\Theta(n)$ to $O(2k\log(n/(2k)))$. This value is in the same order as $M^C_{k,n}$, while the number of links in $\mathcal{G}^4$ is only $2n$, compared with $n(n-1)/2$ links in a complete graph.

Moreover, our estimate $O(2k\log(n/(2k)))$ on the minimum number of measurements required to recover $k$-sparse vectors greatly improves over the existing results in [33, 142], both of which are based on the mixing time of a random walk. The mixing time $T(n)$ can be roughly interpreted as the minimum length of a random walk on graph $G$ such that its distribution is close to the stationary distribution on $G$. Xu *et al.* [142] proved that $O(kT^2(n)\log n)$ measurements can identify $k$-sparse vectors with overwhelming probability by compressed sensing. Cheraghchi *et al.* [33] used $O(k^2T^2(n)\log(n/k))$ measurements to identify $k$ non-zero elements by group testing. In $\mathcal{G}^4$, $T(n)$ should be at least $n/4$. Then both results provide no saving in the number of measurements for $\mathcal{G}^4$ as the mixing time is $\Theta(n)$.

Besides the explicit measurement construction based on the hub idea, we can also recover $k$-sparse vectors associated with $\mathcal{G}^4$ with $O(\log n)$ random measurements. We need to point out that these random measurements do not depend on the measurement constructions for a complete graph.

Consider an $n$-step Markov chain $\{X_k, 1 \leq k \leq n\}$ with $X_1 = 1$. For any $k \leq n-1$, if $X_k = 0$, then $X_{k+1} = 1$; if $X_k = 1$, then $X_{k+1}$ can be 0 or 1 with equal probability. Clearly any realization of this Markov chain does not contain two or more consecutive zeros, and thus is a feasible row of the measurement matrix. Moreover,

**Theorem 12.** *With high probability all k-sparse vectors associated with $\mathcal{G}^4$ can be recovered with $O(g(k)\log n)$ measurements obtained from the above Markov chain, where $g(k)$ is a function of k.*

*Proof.* See Appendix. $\square$

Adding $n$ links in the form $(i, i+2(\text{mod } n))$ to the ring network greatly reduces the number of measurements needed from $\Theta(n)$ to $O(\log n)$. Then how many links in the form $(i, i+2(\text{mod } n))$ shall we add to the ring network such that the minimum number of measurements required to recover $k$-sparse vectors is exactly $\Theta(\log n)$? The answer is $n - \Theta(\log n)$. To see this, let $\mathcal{G}^4_h$ denote the graph obtained by deleting $h$ links in the form $(i, i+2(\text{mod } n))$ from $\mathcal{G}^4$. For example in Fig. 3.4 (b), we delete links $(3, 5)$, $(8, 10)$ and $(9, 11)$ in red dashed lines from $\mathcal{G}^4$. Given $h$, our following results do not depend on the specific choice of links to remove. We have

**Theorem 13.** *The minimum number of measurements required to recover k-sparse vectors associated with $\mathcal{G}^4_h$ is lower bounded by $\lceil h/2 \rceil$, and upper bounded by $2M^C_{k,\lceil \frac{n}{2} \rceil} + h + 2$.*

*Proof.* Let $D$ denote the set of nodes such that for every $i \in D$, link $(i-1, i+1)$ is removed from $\mathcal{G}^4$. The proof of the lower bound follows the proof of Theorem 2 in [121]. The key idea is that recovering one non-zero element in $D$ is equivalent to recovering one non-zero element in a ring network with $h$ nodes, and thus $\lceil h/2 \rceil$ measurements are necessary.

For the upper bound, we first measure nodes in $D$ separately with $h$ measurements. Let $S$ contain the even nodes in $D$ and all the odd nodes. $S$ can be used as a hub to recover the $k$-sparse subvectors associated with the even nodes that are not in $D$, and the number of measurements used is at most $M^C_{k,\lfloor \frac{n}{2} \rfloor} + 1$. We similarly recover $k$-sparse subvectors associated with odd nodes that are not in $D$ using the set of the odd nodes in $D$ and all the even nodes as a hub. The number of measurements is at most $M^C_{k,\lceil \frac{n}{2} \rceil} + 1$. Sum them up and the upper bound follows. $\square$

Together with (3.1), Theorem 13 directly implies that if $\Theta(\log n)$ links in the form $(i, i + 2(\text{mod } n))$ are deleted from $\mathcal{G}^4$, then $\Theta(\log n)$ measurements are both necessary and sufficient to recover associated $k$-sparse vectors for any constant $k$.

Since the number of measurements required by compressed sensing is greatly reduced when we add $n$ links to the ring network, one may wonder whether the number of measurements needed by group testing can be greatly reduced or not. Our next result shows that this is not the case for group testing, please refer to Appendix for its proof.

**Proposition 2.** $\lfloor n/4 \rfloor$ *measurements are necessary to locate two non-zero elements associated with* $\mathcal{G}^4$ *by group testing.*

By Corollary 3 and Proposition 2, we observe that in $\mathcal{G}^4$, with compressed sensing the number of measurements needed to recover $k$-sparse vectors is $O(2k \log(n/(2k)))$, while with group testing, $\Theta(n)$ measurements are required if $k \geq 2$.

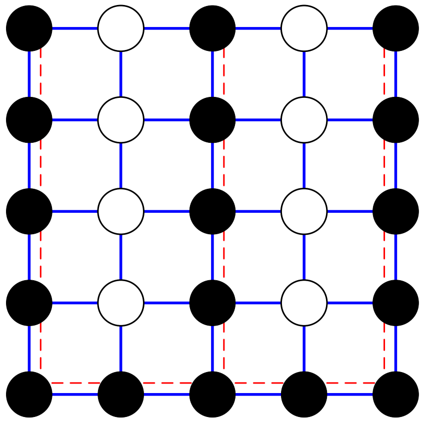


Figure 3.5: Two-dimensional grid

### 3.3.3 Two-dimensional grid

Next we consider the two-dimensional grid, denoted by $\mathcal{G}^{2d}$. $\mathcal{G}^{2d}$ has $\sqrt{n}$ rows and $\sqrt{n}$ columns. We assume $\sqrt{n}$ to be even here, and also skip '$\lceil\cdot\rceil$' and '$\lfloor\cdot\rfloor$' for notational simplicity.

The idea of measurement construction is still the use of a hub. First, Let $S_1$ contain the nodes in the first row and all the nodes in the odd columns, i.e., the black nodes in Fig. 3.5. Then $S_1$ can be used as a hub to measure $k$-sparse subvectors associated with nodes in $V\backslash S_1$. The number of measurements is $M^C_{k,(n/2-\sqrt{n}/2)}+1$. Then let $S_2$ contain the nodes in the first row and all the nodes in the even columns, and use $S_2$ as a hub to recover up to $k$-sparse subvectors associated with nodes in $V\backslash S_2$. Then number of measurements required is also $M^C_{k,(n/2-\sqrt{n}/2)}+1$. Finally, use nodes in the second row as a hub to recover sparse subvectors associated with nodes in the first row. Since nodes in the second row are already identified in the above two steps, then we do not need to measure the hub separately in this step. The number of measurements here is $M^C_{k,\sqrt{n}}$. Therefore,

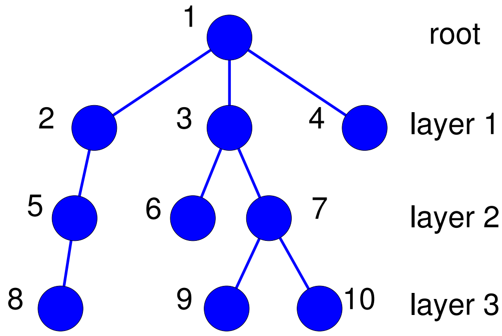



Figure 3.6: Tree topology

*With* $2M^C_{k,n/2-\sqrt{n}/2} + M^C_{k,\sqrt{n}} + 2$ *measurements one can ecover k-sparse vectors associated with* $\mathcal{G}^{2d}$.

### 3.3.4 Tree

Next we consider a tree topology as in Fig. 3.6. For a given tree, the root is treated as the only node in layer 0. The nodes that are $t$ steps away from the root are in layer $t$. We say the tree has depth $h$ if the farthest node is $h$ steps away from the root. Let $n_i$ denote the number of nodes on layer $i$, and $n_0 = 1$. We construct measurements to recover vectors associated with a tree by the following *tree approach*.

We recover the nodes layer by layer starting from the root, and recovering nodes in layer $i$ requires that all the nodes above layer $i$ should already be recovered. First measure the root separately. When recovering the subvector associated with nodes in layer $i$ ($2 \leq i \leq h$), we can measure the sum of any subset of nodes in layer $i$ using some nodes in the upper layers as a hub and then delete the value of the hub from the obtained sum. One simple way to find a hub is to trace back from nodes to be measured on the tree simultaneously until they reach one same node. For example in Fig. 3.6, to measure the sum of nodes 5 and 7, we trace back to the root and measure the sum of nodes 1, 2, 3, 5, and 7 and then subtract the values of nodes 1, 2, and 3, which are already identified when we recover nodes in the upper layers. With this approach, we have,

$\sum_{i=0}^{h} M^C_{k,n_i}$ *measurements are enough to recover k-sparse vectors associated with a tree with depth* h*, where* $n_i$ *is the number of nodes in layer* i*.*

## 3.4 Sparse Recovery over General Graphs

In this section we consider recovering $k$-sparse vectors associated with general graphs. The graph is assumed to be connected. If not, we design measurements to recover $k$-sparse subvectors associated with each component separately.

In Section 3.4.1 we propose a general design guideline based on "$r$-partition". The key idea is to divide the nodes into a small number of groups such that each group can be measured with the help of a hub. Since finding the minimum number of such groups turns out to be NP-hard in general, in Section 3.4.2 we propose a simple algorithm to design measurements on any given graph.

### 3.4.1 Measurement Construction Based on $r$-partition

**Definition 3** ($r$-partition)**.** *Given $G = (V, E)$, disjoint subsets $N_i$ ($i = 1, ..., r$) of $V$ form an $r$-**partition** of $G$ if and only if these two conditions both hold: (1) $\cup_{i=1}^{r} N_i = V$, and*

*(2)* $\forall i$*,* $V \backslash N_i$ *is a hub for* $N_i$*.*

Clearly, $T_o$ and $T_e$ form a 2-partition of graph $\mathcal{G}^4$. With Definition 3 and Theorem 11, we have

**Theorem 14.** *If* $G$ *has an* $r$*-partition* $N_i$ $(i = 1, ..., r)$*, then the number of measurements needed to recover* $k$*-sparse vectors associated with* $G$ *is at most* $\sum_{i=1}^{r} M^C_{k,|N_i|} + r$*, which is* $O(rk \log(n/k))$*.*

Another example of the existence of an $r$-partition is the Erdős-Rényi random graph $G(n, p)$ with $p > \log n/n$. The number of our constructed measurements on $G(n, p)$ is less than the existing estimates in [33, 142]. Please refer to Section 3.5 for the detailed discussion.

Clearly, if an $r$-partition exists, the number of measurements also depends on $r$. In general one wants to reduce $r$ so as to reduce the number of measurements. Given graph $G$ and integer $r$, the question that whether or not $G$ has an $r$-partition is called *r-partition problem*. In fact,

**Theorem 15.** $\forall r \geq 3$*, r-partition problem is NP-complete.*

Please refer to Appendix for its proof. We remark that we cannot prove the hardness of the 2-partition problem though we conjecture it is also a hard problem.

### 3.4.2 Measurement Construction Algorithm for General Graphs

Section 3.4.1 proposes the $r$-partition concept as a measurement design guideline. But finding an $r$-partition with the smallest $r$ in general is NP-hard. Given a connected graph $G$, how shall we efficiently design a small number of measurements to recover $k$-sparse vectors associated with $G$?

One simple way is to find the spanning tree of $G$, and then use the tree approach in Section 3.3.4. The depth of the spanning tree is at least $R$, where $R = \min_{u \in V} \max_{v \in V} d_{uv}$ is the radius of $G$ with $d_{uv}$ as the length of the shortest path between $u$ and $v$. This approach only uses links in the spanning tree, and the number of measurements needed is large when the radius $R$ is large. For example, the radius of $\mathcal{G}^4$ is $n/4$, then the tree approach uses at least $n/4$ measurements, while $O(2k \log(n/2k))$ measurements are already enough if we take advantage of the additional links not in the spanning tree.

Here we propose a simple algorithm to design the measurements for general graphs. The algorithm combines the ideas of the tree approach and the $r$-partition. We still divide nodes into a small number of groups such that each group can be identified via some hub. Here nodes in the same group are the leaf nodes of a spanning tree of a gradually reduced graph. A leaf node has no children on the tree.

Let $G^* = (V^*, E^*)$ denote the orginal graph. The algorithm is built on the following two subroutines. **Leaves**$(G, u)$ returns the set of leaf nodes of a spanning tree of $G$ rooted at $u$. **Reduce**$(G, u, K)$ deletes $u$ from $G$ and fully connects all the neighbors of $u$. Specifically, for every two neighbors $v$ and $w$ of $u$, we add a link $(v, w)$, if not already exist, and let $K_{(v,w)} = K_{(v,u)} \cup K_{(u,w)} \cup \{u\}$, where for each link $(s, t)$, $K_{(s,t)}$ denotes the set of nodes, if any, that connects $s$ and $t$ in the original graph $G^*$. We record $K$ such that measurements constructed on a reduced graph $G$ can be feasible in $G^*$.

**Subroutine 1 Leaves**($G$, $u$)

**Initial:** graph $G$, root $u$

Find a spanning tree $T$ of $G$ rooted at $u$ by breadth-first search, and let $S$ denote the set of leaf nodes of $T$.

**return** $S$

**Subroutine 2 Reduce**($G$, $u$, $K$)

**Initial:** $G = (V, E)$, $H_e$ for each $e \in E$, and node $u$

$V = V \backslash u$.

**for each** two different neighbors $v$ and $w$ of $u$ **do**

  **if** $(v, w) \notin E$ **then**

    $E = E \cup (v, w)$, $K_{(v,w)} = K_{(v,u)} \cup K_{(u,w)} \cup \{u\}$.

  **end if**

**end for**

**return** $G$, $K$

Given graph $G^*$, let $u$ denote the node such that $\max_{v \in V^*} d_{uv} = R$, where $R$ is the radius of $G^*$. Pick $u$ as the root and obtain a spanning tree $T$ of $G^*$ by breadth-first search. Let $S$ denote the set of leaf nodes in $T$. With $V^* \backslash S$ as a hub, we can design $f(k, |S|) + 1$ measurements to recover up to $k$-sparse vectors associated with $S$. We then reduce the network by deleting every node $v$ in $S$ and fully connects its neighbors. For the reduced network $G$, we repeat the above process until all the nodes are deleted. Note that when designing the measurements in a reduced graph $G$, if a measurement passes link $(v, w)$, then it should also include nodes in $K_{(v,w)}$ so as to be feasible in the original graph $G^*$.

In each step tree $T$ is rooted at node $u$ where $\max_{v \in V} d_{uv}$ equals the radius of the current graph $G$. Since all the leaf nodes of $T$ are deleted in the graph reduction procedure, the radius of the new obtained graph should be reduced by at least one. Then we have at most $R$ iterations in Algorithm 1 until only one node is left. Clearly we have,

**Theorem 16.** *The number of measurements designed by Algorithm 1 is at most* $Rf(k, n) + R + 1$, *where* $R$ *is the radius of the graph.*

**Algorithm 1** Measurement construction for graph $G^*$

**Initial:** $G^* = (V^*, E^*)$.

$G = G^*$, $K_e = \emptyset$ for each $e \in E$
**while** $|V| > 1$ **do**
  Find the node $u$ such that $\max_{v \in V} d_{uv} = R^G$, where $R^G$ is the radius of $G$. $S$ =**Leaves**$(G, u)$.
  Design $f(k, |S|) + 1$ measurements to recover $k$-sparse vectors associated with $S$ using nodes in $V \backslash S$ as a hub.
  **for each** $v$ in $S$ **do**
    $G$ = **Reduce**$(G, v, K)$
  **end for**
**end while**
Measure the last node in $V$ directly.
**Output:** All the measurements.

We remark that the number of measurements by the spanning tree approach is also no greater than $Rf(k, n) + R + 1$. However, since Algorithm 1 also considers links that are not in the spanning tree, we expect that for general graphs, it uses fewer measurements than the spanning tree approach. This is verified in Experiment 1 in Section 3.8.

## 3.5 Sparse Recover over Random Graphs

Here we consider measurement constructions over the Erdős-Rényi random graph $G(n, p)$, which has $n$ nodes and every two nodes are connected by a link independently with probability $p$. The behavior of $G(n, p)$ changes significantly when $p$ varies. We study the dependence of number of measurements needed for sparse recovery on $p$.

### 3.5.1 $np = \beta \log n$ for some constant $\beta > 1$

Now $G(n, p)$ is connected almost surely [106]. Moreover, we have the following lemma regarding the existence of an $r$-partition.

**Lemma 9.** *When $p = \beta \log n/n$ for some constant $\beta > 1$, with probability at least $1 - O(n^{-\alpha})$ for some $\alpha > 0$, every set $S$ of nodes with size $|S| = n/(\beta - \epsilon)$ for any $\epsilon \in (0, \beta - 1)$ forms a hub for the complementary set $T = V \backslash S$, which implies that $G(n, p)$ has a $\lceil \frac{\beta-\epsilon}{\beta-\epsilon-1} \rceil$-partition.*

*Proof.* Note that the subgraph $G_S$ is also Erdős-Rényi random graph in $G(n/(\beta - \epsilon), p)$. Since $p = \beta \log n/n > \log(n/(\beta-\epsilon))/(n/(\beta-\epsilon))$, $G_S$ is connected almost surely.

Let $P_f$ denote the probability that there exists some $u \in T$ such that $(u, v) \notin E$ for every $v \in S$. Then

$$\begin{aligned} P_f &= \sum_{u \in T} (1-p)^{|S|} = (1 - \frac{1}{\beta - \epsilon}) n (1 - \beta \log n/n)^{n/(\beta-\epsilon)} \\ &= (1 - 1/(\beta - \epsilon)) n (1 - \beta \log n/n)^{\frac{n}{\beta \log n} \cdot \frac{\beta \log n}{\beta - \epsilon}} \\ &\leq (1 - \frac{1}{\beta - \epsilon}) n e^{-\frac{\beta \log n}{\beta - \epsilon}} \quad \leq (1 - \frac{1}{\beta - \epsilon}) n^{-\epsilon/(\beta-\epsilon)}. \end{aligned}$$

Thus, $S$ is a hub for $T$ with probability at least $1 - O(n^{-\alpha})$ for $\alpha = \epsilon/(\beta - \epsilon) > 0$. Since the size of $T$ is $(1 - 1/(\beta - \epsilon))n$, $G(n, p)$ has at most $\lceil \frac{\beta-\epsilon}{\beta-\epsilon-1} \rceil$ such disjoint sets. Then by a simple union bound, one can conclude that $G(n, p)$ has a $\lceil \frac{\beta-\epsilon}{\beta-\epsilon-1} \rceil$-partition with probability at least $1 - O(n^{-\alpha})$. □

For example, when $\beta > 2$, Lemma 9 implies that any two disjoint sets $N_1$ and $N_2$ with $|N_1| = |N_2| = n/2$ form a 2-partition of $G(n, p)$ with probability $1 - O(n^{-\alpha})$. From Theorem 14 and Lemma 9, and let $\epsilon \to 0$, we have

*When $p = \beta \log n/n$ for some constant $\beta > 1$, all k-sparse vectors associated with $G(n, p)$ can be identified with $O(\lceil \frac{\beta}{\beta-1} \rceil k \log(n/k))$ measurements with probability at least $1 - O(n^{-\alpha})$ for some $\alpha > 0$.*

[33] considers group testing over Erdős-Rényi random graphs and shows that $O(k^2 \log^3 n)$ measurements are enough to identify up to $k$ non-zero entries if it further holds that $p = \Theta(k \log^2 n/n)$. Here with compressed sensing setup and $r$-partition results, we can recover $k$-sparse vectors in $\mathcal{R}^n$ with $O(k \log(n/k))$ measurements when $p > \log n/n$. This result also improves over the previous result in [142], which requires $O(k \log^3 n)$ measurements for compressed sensing on $G(n, p)$.

### 3.5.2 $np - \log n \to +\infty$, and $\frac{np - \log n}{\log n} \to 0$

Roughly speaking, $p$ is just large enough to guarantee that $G(n, p)$ is connected almost surely [106]. The diameter $D = \max_{u,v} d_{uv}$ of a connected graph is the greatest distance between any pair of nodes, and here it is concentrated around $\frac{\log n}{\log \log n}$ almost surely [14]. We design measurements on $G(n, p)$ with Algorithm 1. With Theorem 16 and the fact that the radius $R$ is no greater than the diameter $D$ by definition, we have

*When $np - \log n \to +\infty$, and $\frac{np - \log n}{\log n} \to 0$, $O(k \log n \log(n/k)/ \log \log n)$ measurements can identify k-sparse vectors associated with $G(n, p)$ almost surely.*

### 3.5.3 $1 < c = np < \log n$

Now $G(n, p)$ is disconnected and has a unique giant component containing $(\alpha + o(1))n$ nodes almost surely with $\alpha$ satisfying $e^{-c\alpha} = 1 - \alpha$, or equivalently,

$$\alpha = 1 - \frac{1}{c} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (ce^{-c})^k,$$

and all the other nodes belong to small components. The expectation of the total number of components in $G(n, p)$ is $(1 - \alpha - c(1 - \alpha)^2/2 + o(1))n$ [106]. Since it is necessary to take at least one measurement for each component, $(1 - \alpha - c(1 - \alpha)^2/2 + o(1))n$ is an expected lower bound of measurements required to identify sparse vectors.

The diameter $D$ of a disconnected graph is defined to be the largest distance between any pair of nodes that belong to the same component. Since $D$ is now $\Theta(\log n/\log(np))$ almost surely [34], then for the radius $R$ of the giant component, $R \leq D = O(\log n/\log(np))$, where the second equality holds almost surely. We use Algorithm 1 to design measurements on the giant component, and then measure every node in the small components directly. Thus, $k$-sparse vectors associated with $G(n, p)$ can be identified almost surely with $O(k \log n \log(n/k)/\log(np)) + (1 - \alpha + o(1))n$ measurements.

Note that here almost surely the size of every small component is at most $\frac{\log n + 2\sqrt{\log n}}{np - 1 - \log(np)}$ (Lemma 5, [34]). If $k = \Omega(\log n)$, almost surely $(1 - \alpha + o(1))n$ measurements are necessary to identify subvectors associated with small components, and thus necessary for identifying $k$-sparse vectors associated with $G(n, p)$. Combing the arguments, we have

*When $1 < c = np < \log n$ with constant $c$, we can identify $k$-sparse vectors associated with $G(n, p)$ almost surely with $O(k \log n \log(n/k)/\log(np)) + (1 - \alpha + o(1))n$*

*measurements.* $(1 - \alpha - c(1-\alpha)^2/2 + o(1))n$ *is an expected lower bound of the number of measurements needed. Moreover, if* $k = \Omega(\log n)$*, almost surely* $(1 - \alpha + o(1))n$ *measurements are necessary to identify k-sparse vectors.*

### 3.5.4 $np < 1$

Since the expectation of the total number of components in $G(n, p)$ with $np < 1$ is $n - pn^2/2 + O(1)$ [106], then $n - pn^2/2 + O(1)$ is an expected lower bound of the number of measurements required. Since almost surely all components are of size $O(\log n)$, then we need to take $n$ measurements when $k = \Omega(\log n)$. Therefore,

*When* $np < 1$*, we need at least* $n - pn^2/2 + O(1)$ *measurements to identify k-sparse vectors associated with* $G(n, p)$ *in expectation. Moreover, when* $k = \Omega(\log n)$*,* $n$ *measurements are necessary almost surely.*

## 3.6 Adding additional graph constraints

Our constructions are based on assumptions (A1) and (A2). Here we consider additional graph constraints brought by practical implementation. We first consider measurement construction with length constraint, as measurements with short length are preferred in practice. We then discuss the scenario when only a subset of nodes can act as agents and each measurement should pass at least one agent.

### 3.6.1 Measurements with short length

We have not imposed any constraint on the number of nodes in one measurement. In practice, one may want to take short measurements so as to reduce the communication cost and the measurement noise. We next consider sparse recovery with additional constraint on measurement length, and we discuss two special graphs.

#### Line and Ring

The construction in Section 3.3.1 is optimal for a line network in terms of the number of measurements needed, and the length of each measurement is $\lfloor \frac{n+1}{k+1} \rfloor$, which is proportional to $n$ when $k$ is a constant. Here we provide a different construction such that the total number of measurements needed to recover associated $k$-sparse vectors is $k\lceil \frac{n}{k+1} \rceil + 1$, but each measurement measures at most $k+2$ nodes. We also remark that the number of measurements by this construction is within the minimum plus $\max(k-1, 1)$ for a line network, and the minimum plus $k$ for a ring network.

We construct the measurements as follows. Given $k$, let $B^k$ be a $k+1$ by $k+1$ square matrix with entries of '1' on the main diagonal and the first row, i.e. $B^k_{ii} = 1$ and $B^k_{1i} = 1$ for all $i$. If $k$ is even, let $B^k_{i(i-1)} = 1$ for all $2 \leq i \leq k+1$; if $k$ is odd, let $B^k_{i(i-1)} = 1$ for all $2 \leq i \leq k$. $B^k_{ij} = 0$ elsewhere. Let $t = \lceil \frac{n}{k+1} \rceil$, we construct a $(kt+1)$ by $(k+1)t$ matrix $A$ based on $B^k$. Given set $S \subseteq \{1, ..., kt+1\}$ and set $T \subseteq \{1, ..., (k+1)t\}$, $A_{ST}$ is the submatrix of $A$ with row indices in $S$ and column indices in $T$. For all $i = 1, ..., t$, let $S_i = \{(i-1)k+1, ..., ik+1\}$, and let $T_i = \{(k+1)(i-1)+1, ..., (k+1)i\}$. Define $A_{S_iT_i} = B^k$ for all $i$. All the other entries of $A$ are zeros. We keep the first $n$ columns of $A$ as a measurement matrix for the line/ring network with $n$ nodes. Note that the last one or serval rows of the reduced matrix can be all zeros, and we just delete such rows, and let the resulting matrix be the measurement matrix. For example, when $k = 2$ and $n = 9$, we have $t = 3$, and

$$B^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

and

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}. \tag{3.5}$$

When $k = 3$, and $n = 8$, we have $t = 2$ and

$$B^3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Each measurement measures at most $k + 2$ nodes when $k$ is even and at most $k + 1$ nodes when $k$ is odd. We have,

**Theorem 17.** *The above construction can recover $k$-sparse vectors associated with a line/ring network with at most $k\lceil \frac{n}{k+1} \rceil + 1$ measurements, which is within the minimum number of measurements needed plus $k$. And each measurement measures at most $k+2$ nodes.*

*Proof.* We only need to prove that all $k$-sparse vectors in $\mathcal{R}^{(k+1)t}$ can be identified with $A$, which happens if and only if for every vector $\mathbf{z} \neq \mathbf{0}$ such that $A\mathbf{z} = \mathbf{0}$, $\mathbf{z}$ has at least $2k+1$ non-zero elements.

If $t = 1$, $A$ a $k+1$ by $k+1$ full rank matrix, and the claim holds trivially. We next consider $t \geq 2$. We prove the case when $k$ is even, and skip the similar proof for odd $k$.

For each integer $t'$ in $[2, t]$, define a submatrix $A_{t'}$ formed by the first $kt' + 1$ rows and the first $(k+1)t'$ columns of $A$. For example, for $A$ in (3.5), we define

$$A_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \text{and } A_3 = A.$$

We will prove by induction on $t'$ that (#) every non-zero vector $\mathbf{z} \in \mathcal{R}^{(k+1)t'}$ such that $A_{t'}\mathbf{z} = \mathbf{0}$ holds has at least $2k+1$ non-zero elements for every $t'$ in $[2, t]$.

First consider $A_2$, which is a $(2k+1) \times (2k+2)$ matrix. From the last $k$ rows of $A_2$, one can easily argue that for every $\mathbf{z}$ such that $A_2\mathbf{z} = \mathbf{0}$, its last $k+1$ entries are either all zeros or all non-zeros. If the last $k+1$ entries of $\mathbf{z}$ are all zeros, let $\mathbf{z}'$ denote the subvector containing the first $k+1$ entries of $\mathbf{z}$. Then we have $\mathbf{0} = A_2\mathbf{z} = B^k\mathbf{z}'$. Since $B^k$ is full rank, then $\mathbf{z}' = \mathbf{0}$, which implies that $\mathbf{z} = \mathbf{0}$.

Now consider the case that last $k+1$ entries of $\mathbf{z}$ are all non-zeros. Since $k+1$ is odd, the sum of these entries is,

$$\sum_{i=k+2}^{2k+2} z_i = z_{k+2} \neq 0. \tag{3.6}$$

Let $\mathbf{a}_i^T$ $(i = 1, ..., 2k+1)$ denote the $i$th row of $A_2$. We have

$$\mathbf{a}_{k+1}^T \mathbf{z} = \sum_{i=k}^{2k+2} z_i = 0. \tag{3.7}$$

Combining (3.6) and (3.7), we know that

$$z_k + z_{k+1} = -z_{k+2} \neq 0. \tag{3.8}$$

Thus, at least one of $z_k$ and $z_{k+1}$ is non-zero. Combining (3.8) with $\mathbf{a}_1^T \mathbf{z} = 0$, we have one of the first $k-1$ entries of $\mathbf{z}$ is non-zero. From $\mathbf{a}_i^T \mathbf{z} = 0$ for $2 \leq i \leq k-1$, one can argue that if one of the first $k-1$ entries of $\mathbf{z}$ is non-zero, then all the first $k-1$ entries are non-zero. Therefore, $\mathbf{z}$ has at least $2k+1$ nonzero entries. (#) holds for $A_2$.

Now suppose (#) holds for some $t'$ in $[2, t-1]$. Consider matrix $A_{t'+1}$. Same as the arguments for $A_2$, one can show that for every $\mathbf{z} \neq \mathbf{0}$ such that $A_{t'+1}\mathbf{z} = \mathbf{0}$, its last $k+1$ entries are either all zeros or all non-zero. In the former case, let $\mathbf{z}'$ denote the subvector containing the first $(k+1)t'$ entries of $\mathbf{z}$. By induction hypothesis, $\mathbf{z}'$ has at least $2k+1$ nonzero entries, thus so does $\mathbf{z}$.

If the last $k+1$ entries of $\mathbf{z}$ are all non-zero, like in the $A_2$ case, we argue that the sum of $z_{(k+1)t'-1}$ and $z_{(k+1)t'}$ is non-zero, which implies that at least one of them is non-zero. Also consider $\mathbf{a}_i^T \mathbf{z} = 0$ with $i = rk+1$ for every integer $r$ in $[0, t'-1]$, one can argue that there exist $j$ in $[0, t'-1]$ such that the sum of all $k-1$ entries from $z_{j(k+1)+1}$ to $z_{j(k+1)+k-1}$ is non-zero. Then, from $\mathbf{a}_i^T \mathbf{z} = 0$ for $i = jk+2, ..., jk+k-1$, we know that if the sum of $z_{j(k+1)+1}$ to $z_{j(k+1)+k-1}$ is non-zero, every entry is non-zero. We conclude that in this case $\mathbf{z}$ also has at least $2k + 1$ nonzero entries.

By induction over $t'$, every $\mathbf{z} \neq \mathbf{0}$ such that $A\mathbf{z} = \mathbf{0}$ has at least $2k + 1$ non-zero entries, then the result follows. □

This construction measures at most $k+2$ nodes in each measurement. If measurements with constant length are preferred, we provide another construction method such that every measurement only measures at most three nodes. This method requires more measurements, $(2k - 1)\lceil \frac{n}{2k} \rceil + 1$ measurements to recover $k$-sparse vectors associated with a line/ring network.

Given $k$, let $D^k$ be a $2k$ by $2k$ square matrix having entries of '1' on the main diagonal and the subdiagonal and '0' elsewhere, i.e. $D^k_{ii} = 1$ for all $i$ and $D^k_{i(i-1)} = 1$ for all $i \geq 2$, and $D^k_{ij} = 0$ elsewhere. Let $t = \lceil \frac{n}{2k} \rceil$, we construct a $(2kt - t + 1)$ by $2kt$ matrix $A$ based on $D^k$. Let $S_i = \{(i - 1)(2k - 1) + 1, ..., i(2k - 1) + 1\}$, and let $T_i = \{2k(i-1)+1, ..., 2ki\}$. Define $A_{S_i T_i} = D^k$ for all $i = 1, ..., t$, and $A_{ij} = 0$ elsewhere. We keep the first $n$ columns of $A$ as the measurement matrix. For example, when $k = 2$ and $n = 8$, we have

$$D^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}. \tag{3.9}$$

**Theorem 18.** *The above constructed* $(2k-1)\lceil \frac{n}{2k} \rceil + 1$ *measurements can identify k-sparse vectors associated with a line/ring network of n nodes, and each measurement measures at most three nodes.*

*Proof.* When $t = 1$, $A$ is a full rank square matrix. We focus on the case that $t \geq 2$. For each integer $t'$ in $[2, t]$, define a submatrix $A_{t'}$ formed by the first $2kt' - t' + 1$ rows and the first $2kt'$ columns of $A$. We will prove by induction on $t'$ that every $\mathbf{z} \neq \mathbf{0}$ such that $A_{t'}\mathbf{z} = \mathbf{0}$ holds has at least $2k + 1$ non-zero elements for every $t'$ in $[2, t]$.

First consider $A_2$. For $A$ in (3.9), $A_2 = A$. From the first $2k - 1$ rows of $A_2$, one can check that for every $\mathbf{z}$ such that $A_2\mathbf{z} = \mathbf{0}$, its first $2k - 1$ entries are zeros. From the $2k$th row of $A_2$, we know that $z_{2k}$ and $z_{2k+1}$ are either both zeros or both non-zero. In the former case, the remaining $2k - 1$ entries of $\mathbf{z}$ must be zeros, thus, $\mathbf{z} = \mathbf{0}$. In the latter case, one can check that the remaining $2k - 1$ entries are all non-zero, and therefore $\mathbf{z}$ has $2k + 1$ non-zero entries.

Now suppose the claim holds for some $t'$ in $[2, t - 1]$. Consider vector $\mathbf{z} \neq \mathbf{0}$ such that $A_{t'+1}\mathbf{z} = \mathbf{0}$. If $z_{2kt'+1} = 0$, it is easy to see that the last $2k$ entries of $\mathbf{z}$ are all zeros. Then by induction hypothesis, at least $2k + 1$ entries of the first $2kt'$

elements of $\mathbf{z}$ are non-zero. If $z_{2kt'+1} \neq 0$, one can check that the last $2k-1$ entries of $\mathbf{z}$ are all non-zero, and at least one of $z_{2kt'-1}$ and $z_{2kt'}$ is non-zero. Thus, $\mathbf{z}$ also has at least $2k+1$ non-zero entries in this case.

By induction over $t'$, every $\mathbf{z} \neq \mathbf{0}$ such that $A\mathbf{z} = \mathbf{0}$ has at least $2k+1$ non-zero entries, then the theorem follows. $\square$

The number of measurements by this construction is greater than those of the previous methods. But the advantage of this construction is that the number of nodes in each measurement is at most three, no matter how large $n$ and $k$ is.

**Ring with each node connecting to four neighbors**

We next consider $\mathcal{G}^4$ in Fig. 3.4 (a). We further impose the constraint that the number of nodes in each measurement cannot exceed $d$ for some predetermined integer $d$. We neglect $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ for notational simplicity.

All the even nodes are divided into $n/d$ groups, each of which contains $d/2$ consecutive even nodes. We use each group as a hub to measure $d/2$ odd nodes that have direct links with nodes in the hub. Then we can identify the values related to all the odd nodes with $nM^C_{k,d/2}/d + n/d$ measurements, and the number of nodes in each measurement does not exceed $d$. We then measure the even nodes with groups of odd nodes as hubs. In total, the number of measurements is $2nM^C_{k,d/2}/d + 2n/d$, which is $O(2kn\log(d/2)/d)$. When $d$ equals to $n$, the result coincides with Theorem 3. Since $n/d$ measurements are needed to measure all the nodes at least once, we have

**Theorem 19.** *The number of measurements needed to recover k-sparse vectors associ-*

*ated with $\mathcal{G}^4$ with each measurement containing at most $d$ nodes is lower bounded by $n/d$, and upper bounded by $O(2kn\log(d/2)/d)$.*

The ratio of the number of measurements by our construction to the minimum number needed with length constraint is within $Ck\log(d/2)$ for some constant $C$.

### 3.6.2 Measurements passing at least one node in a fixed subset

Recall that for a connected subgraph $G_S$, agent $u$ in $S$ aggregates the sum of all the nodes in $S$, and sends the measurement to the central operator. One may want to reduce the number of agents in a network so as to reduce the monitoring and transmission costs. If only a small set $Y$ of nodes can perform as the agents, every constructed measurement is then required to pass at least one node in $Y$.

This can be achieved with small modification to the construction algorithm. Let $Y$ denote the set of agents. Let $D$ denote the set of nodes that can be measured together via a hub $H$. If $H \cap Y$ is not empty, every measurement contains at least one node in $Y$ automatically as all the nodes in $H$ are included. Now consider the case that $H \cap Y$ is empty. If there exists a path $P$ from some node $j$ in $H$ to some node $f$ in $Y$ such that $P \cap D$ is empty, then let $\hat{H} := H \cup P$ be the new hub, and design measurements for $D$ using hub $\hat{H}$. Then every measurement contains all nodes in $\hat{H}$ and thus the agent $f$. If such a path does not exist, pick any node $i$ in $D$ and any node $f$ in $Y$, find the shortest path $P'$ between $i$ and $f$. Let $H' := H \cup P'$ be the hub, and let $D' := D \backslash i$ be the set of nodes that can be measured via $H'$. Then every measurement contains agent $f$. Node $i$ can be measured by two additional measurements, one measures path $P'$, and the

**Subroutine 3 Agent**($H$, $D$, $Y$, $G$)

**Initial:** hub $H$, set $D$ of nodes, set $Y$ of agents, $G$

**if** $H \cap Y \neq \Phi$ **then**
  Design $f(k, |D|) + 1$ measurements to recover $k$-sparse vectors associated with $D$ using $H$ as a hub.
**else**
  Find the shortest path between every node in $H$ and every node in $Y$.
  **if** there exists a shortest path $P$ s.t. $P \cap D = \Phi$ **then**
    Design $f(k, |D|) + 1$ measurements to recover nodes in $D$ using $\hat{H} = H \cup P$ as a hub.
  **else**
    pick a node $i$ in $D$ and a node $f$ in $Y$, find the shortest path $P'$ between $i$ and $f$.
    $D' := D \backslash i$, $H' := H \cup P'$, design $f(k, |D'|) + 1$ measurements to recover $D'$ with $H'$ as a hub.
    Measure $P'$ and $P' \backslash i$ to recover node $i$.
  **end if**
**end if**

other measures $P' \backslash i$. Therefore, with this simple modification, we can measure the same set of nodes with each measurement contains at least one node in $Y$, and the total number of measurements increases by at most two.

We summarize the above modification in subroutine **Agent**. For measurement design on general graphs, we first replace step 4 in Algorithm 1 in Section 3.4.2 with subroutine Agent($V \backslash S$, $S$, $Y$, $G$). Then in each iteration the number of measurements is increased by at most two. We then replace step 9 with measuring the paths $P^*$ and $P^* \backslash n_{\text{last}}$, where $n_{\text{last}}$ is the last node in $G$, and $P^*$ connects $n_{\text{last}}$ to any node $j$ in $Y$ on the original graph. Therefore, the total number of measurements needed by the modified algorithm is upper bounded by $Rf(k, n) + 3R + 2$, and each measurement in the modified version contains at least one node in $Y$.

## 3.7 Sensitivity to hub measurement errors

In constructions based on the use of a hub, in order to measure nodes in $S$ using hub $H$, we first measure the sum of nodes in $H$, and then delete it from other measurements to obtain the sum of some subset of nodes in $S$. This arises the issue that if the sum of $H$ is not measured correctly, this single error would be introduced into all the measurements. Here we prove that successful recovery is still achievable when a hub measurement is erroneous.

Mathematically, let $\mathbf{x}_S$ denote the sparse vector associated with $S$, and let $\mathbf{x}_H$ denote the vector associated with $H$ and let $A^{m\times|S|}$ be a measurement matrix that can identify $k$-sparse vectors associated with a complete graph of $|S|$ nodes. We arrange the vector $\mathbf{x}$ such that $\mathbf{x} = [\mathbf{x}_S^T \quad \mathbf{x}_H^T]^T$, then

$$F = \begin{bmatrix} A & W^{m\times|H|} \\ \mathbf{0}_{|S|}^T & \mathbf{1}_{|H|}^T \end{bmatrix}$$

is the measurement matrix for detecting $k$ non-zeros in $S$ using hub $H$, where $W$ is a matrix with all '1's, $\mathbf{0}$ is a column vector of all '0's, and $\mathbf{1}$ is a column vector of all '1's. Let vector $\mathbf{z}$ denote the first $m$ measurements, and let $z_0$ denote the last measurement of the hub $H$. Then

$$\begin{bmatrix} \mathbf{z} \\ z_0 \end{bmatrix} = \begin{bmatrix} A\mathbf{x}_S + \mathbf{1}^T\mathbf{x}_H\mathbf{1}_m \\ \mathbf{1}^T\mathbf{x}_H \end{bmatrix},$$

or equivalently

$$\mathbf{z} - z_0\mathbf{1}_m = A\mathbf{x}_S. \tag{3.10}$$

If there is some error $e_0$ in the last measurement, i.e., instead of $z_0$, the actual measurement we obtain is

$$\hat{z}_0 = \mathbf{1}^T\mathbf{x}_H + e_0,$$

$e_0$ hurts the recovery accuracy of $\mathbf{x}_S$ through (3.10).

To eliminate the impact of $e_0$, we model it as an entry of an augmented sparse signal to recover. Let $\mathbf{x}' = [\mathbf{x}^T \quad e_0]^T$, and $A' = [A \quad -\mathbf{1}_m]$, we have

$$A'\mathbf{x}' = \mathbf{z} - \hat{z}_0\mathbf{1}_m. \tag{3.11}$$

Then, recovering $\mathbf{x}_S$ in the presence of hub error $e_0$ is equivalent to recovering $k+1$-sparse vector $\mathbf{x}'$ from (3.11).

We consider one special construction of matrix $A^{m\times|S|}$ for a complete graph. $A$ has '1' on every entry in the last row, and takes value '1' and '0' with equal probability independently for every other entry. $A' = [A \quad -\mathbf{1}_m]$, let $\hat{A}$ be the submatrix of the first $m-1$ rows of $A'$. Let $\mathbf{y} = \mathbf{z} - \hat{z}_0\mathbf{1}_m$, and let $\hat{\mathbf{y}}$ denote the first $m-1$ entries of $\mathbf{y}$. We have,

$$(2\hat{A} - W^{(m-1)\times|S|})\mathbf{x}' = 2\hat{\mathbf{y}} - y_m.$$

We recover $\mathbf{x}'$ by solving the $\ell_1$-minimization problem,

$$\min \|\mathbf{x}\|_1, \quad \text{s.t. } (2\hat{A} - W^{(m-1)\times|S|})\mathbf{x} = 2\hat{\mathbf{y}} - y_m. \tag{3.12}$$

**Theorem 20.** *With the above construction of $A$, when $m \geq C(k+1)\log|S|$ for some constant $C > 0$ and $|S|$ is large enough, with probability at least $1 - O(|S|^{-\alpha})$ for some constant $\alpha > 0$, $\mathbf{x}'$ is the unique solution to (3.12) for all $k+1$-sparse vectors $\mathbf{x}'$ in $\mathcal{R}^{|S|+1}$.*

Theorem 20 indicates that even though the hub measurement is erroneous, one can still identify $k$-sparse vectors associated with $S$ with $O((k+1)\log|S|)$ measurements.

The proof of Theorem 20 relies heavily on Lemma 10.

**Lemma 10.** *If matrix $\Phi^{p\times n}$ takes value $-1/\sqrt{p}$ on every entry in the last column and takes value $\pm 1/\sqrt{p}$ with equal probability independently on every other entry, then for any $\delta > 0$, there exists some constant $C$ such that when $p \geq C(k+1)\log n$ and $n$ is large enough, with probability at least $1 - O(n^{-\alpha})$ for some constant $\alpha > 0$ it holds that for every $U \subseteq \{1, ..., n\}$ with $|U| \leq 2k+2$ and for every $\mathbf{x} \in \mathcal{R}^{2k+2}$,*

$$(1-\delta)\|\mathbf{x}\|_2^2 \leq \|\Phi_U \mathbf{x}\|_2^2 \leq (1+\delta)\|\mathbf{x}\|_2^2, \tag{3.13}$$

*where $\Phi_U$ is the submatrix of $\Phi$ with column indices in $U$.*

*Proof.* Consider matrix $\Phi'^{p\times n}$ with each entry taking value $\pm 1/\sqrt{p}$ with equal probability independently. For every realization of matrix $\Phi'$, construct a matrix $\hat{\Phi}$ as follows. For every $i \in \{1, ..., p\}$ such that $\Phi'_{in} = 1/\sqrt{p}$, let $\hat{\Phi}_{ij} = -\Phi'_{ij}$ for all $j = 1, ..., n$. Let $\hat{\Phi}_{ij} = \Phi'_{ij}$ for every other entry. One can check that $\hat{\Phi}$ and $\Phi$ follow the same probability distribution. Besides, according to the construction of $\hat{\Phi}$, for any subset $U \subseteq \{1, ..., n\}$,

$${\Phi'_U}^T \Phi'_U = \hat{\Phi}_U^T \hat{\Phi}_U. \tag{3.14}$$

The Restricted Isometry Property [20] indicates that the statement in Lemma 10 holds for $\Phi'$. From (3.14), and the fact that $\|\Phi'_U \mathbf{x}\|_2^2 = \mathbf{x}^T {\Phi'_U}^T \Phi'_U \mathbf{x}$, the statement also holds for $\hat{\Phi}$. Since $\hat{\Phi}$ and $\Phi$ follow the same probability distribution, the lemma follows. □

*Proof.* (of Theorem 20) From Lemma 10, when $m \geq C(k+1)\log|S|$ for some $C > 0$ and $|S|$ is large enough, with probability at least $1 - O(|S|^{-\alpha})$, matrix $(2\hat{A} - W^{(m-1)\times|S|})/\sqrt{m-1}$ satisfies (3.13) for some small enough $\delta$, say $\delta < \sqrt{2} - 1$. Then from [21, 68], (3.12) can recover all $k+1$-sparse vectors correctly. □

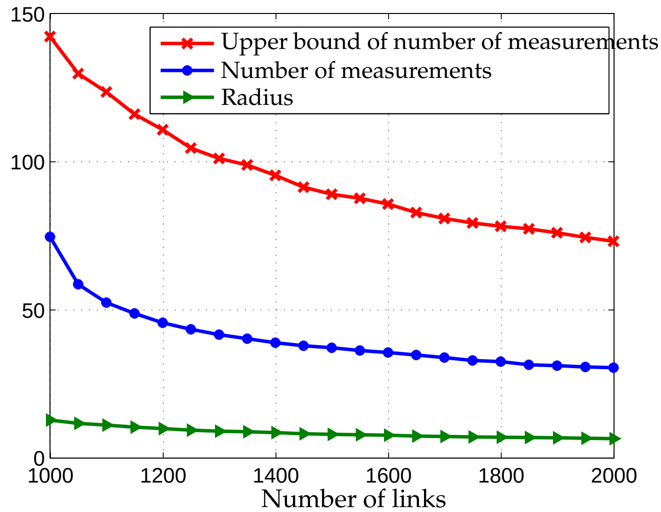



Figure 3.7: Random graph with $n = 1000$

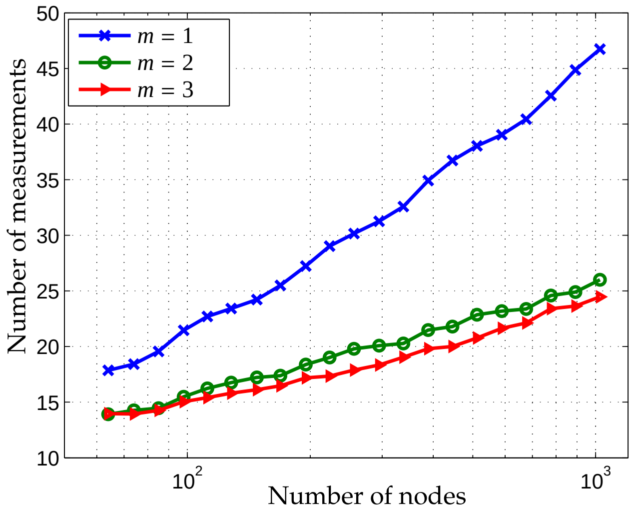



Figure 3.8: BA model with increasing $n$ and different $m$

## 3.8 Simulation

**Experiment 1 (Effectiveness of Algorithm 1):** Given a graph $G$, we apply Algorithm 1 to divide the nodes into groups such that each group (except the last one) can be measured via some hub. The last group contains one node and can

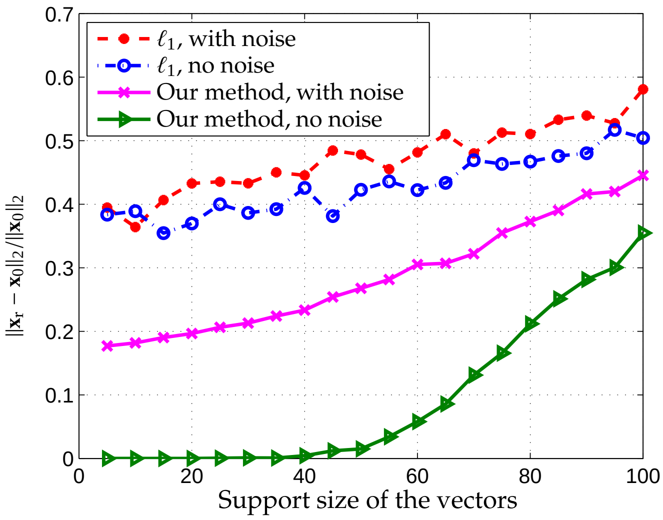



Figure 3.9: Recovery performance with hub errors

be measured directly. It is know that $M^C_{1,n} = \lceil \log(n + 1) \rceil$, and the corresponding measurement matrix has the binary expansion of integer $i$ as column $i$ [57]. Also from (3.1) the number of measurements required to recovery $k$-sparse vectors is within a constant times $kM^C_{1,n}$. Therefore, here we design measurements to recover 1-sparse vectors on $G$ as an example. The total number of constructed measurements is $\sum_i^{q-1} \lceil \log(n_i + 1) \rceil + q$, where $n_i$ is the number of nodes in group $i$ and $q$ is the total number of groups.

In Fig. 3.7, we gradually increase the number of links in a graph with $n = 1000$ nodes. We start with a uniformly generated random tree, and in each step randomly add 25 links to the graph. All the results are averaged over one hundred realizations. The number of measurements constructed decreases from 73 to 30 when the number of links increases from $n - 1$ to $2n - 1$. Note that the number of measurements is already within $3M^C_{1,n}$ when the average node degree is close to 4. The radius $R$ of the graph decreases from 13 to 7, and we also plot the upper bound $R\lceil \log n \rceil + R + 1$ provided by Theorem 16. One can see that the number of measurements actually constructed is much less than the upper bound.

In Fig. 3.8, we consider the scale-free network with Barabási-Albert (BA) model [6] where the graph initially has $m_0$ connected nodes, and each new node connects to $m$ existing nodes with a probability that is proportional to the degree of the existing nodes. We start with a random tree of 10 nodes and increase the total number of nodes from 64 to 1024. Every result is averaged over one hundred realizations. Since the diameter of BA model is $O(\log n/\log\log n)$ [15], then by Theorem 16, the number of our constructed measurements is upper bounded by $O(\log^2 n/\log\log n)$. As the mixing time of BA model is $O(\log n)$ [96], methods in [33] and [142] require $O(\log^3 n)$ random measurements.

**Experiment 2 (Recovery Performance with Hub Error):** We generate a graph with $n = 500$ nodes from BA model. Algorithm 1 divides nodes into four groups with 375, 122, 2 and 1 node respectively. For each of the first two groups with size $n_i$ ($i = 1, 2$), we generate $\lceil n_i/2 \rceil$ random measurements each measuring a random subset of the group together with its hub. Every node of the group is included in the random subset independently with probability 0.5. We also measure the two hubs directly. Each of the three nodes in the next two groups is measured directly by one measurement. The generated matrix $A$ is 254 by 500. We generate a sparse vector $\mathbf{x}_0$ with i.i.d. zero-mean Gaussian entries on a randomly chosen support, and normalize $\|\mathbf{x}_0\|_2$ to 1.

To recover $\mathbf{x}_0$ from $\mathbf{y} = A\mathbf{x}_0$, one can run the widely used $\ell_1$-minimization [21] to recover the subvectors associated with the first two groups, and the last three entries of $\mathbf{x}_0$ can be obtained from measurements directly. However, as discussed in Section 3.7, an error in a hub measurement degrades the recovery accuracy of subvectors associated with that group. To address this issue, we use a modified $\ell_1$-minimization in which the errors in the two hubs are treated as entries of an augmented vector to recover. Specifically, let the augmented vector $\mathbf{z} = [\mathbf{x}_0^T, e_1, e_2]^T$ and the augmented matrix $\tilde{A} = [A\, \boldsymbol{\beta}\, \boldsymbol{\gamma}]$, where $e_1$ (or $e_2$) denotes the error in the measurement of the first (second) hub, and the column vector $\boldsymbol{\beta}$ (or $\boldsymbol{\gamma}$) has '1' in the row corresponding to the measurement of the first (or second) hub and '0' elsewhere. We then recover $\mathbf{z}$ (and thus $\mathbf{x}_0$) from $\mathbf{y} = \tilde{A}\mathbf{z}$ by running $\ell_1$-minimization on each group separately.

Fig. 3.9 compares the recovery performance of our modified $\ell_1$-minimization and the conventional $\ell_1$-minimization, where the hub errors $e_1$ and $e_2$ are drawn from standard Gaussian distribution with zero mean and unit variance. For every support size $k$, we randomly generate two hundred $k$-sparse vectors $\mathbf{x}_0$, and let $\mathbf{x}_\mathrm{r}$ denote the recovered vector. Even with the hub errors, the average $\|\mathbf{x}_\mathrm{r} - \mathbf{x}_0\|_2/\|\mathbf{x}_0\|_2$ is within $10^{-6}$ when $\mathbf{x}_0$ is at most 35-sparse by our method, while by $\ell_1$-minimization, the value is at least 0.35. We also consider the case that besides errors in hub measurements, every other measurement has i.i.d. Gaussian noise with zero mean and variance $0.04^2$. The average $\|\mathbf{x}_\mathrm{r} - \mathbf{x}_0\|_2/\|\mathbf{x}_0\|_2$ here is smaller with our method than that with $\ell_1$-minimization.

CHAPTER 4

# SPARSE RECOVERY WITH NONNEGATIVE SIGNALS

In many applications, the sparse signal of interest may only contain nonnegative entries. Besides the fact that it can be identified from a small number of measurements as indicated by sparse recovery theory, a nonnegative sparse signal may even be the only nonnegative signal satisfying the linear observations. In this chapter we investigate the uniqueness property of a nonnegative vector solution to an underdetermined linear system.

## 4.1 Introduction

Compressed sensing aims to recover sparse signals from an incomplete set of linear observations. In many applications, the vector to recover is nonnegative [16, 55, 146], e.g., transmission delays in the Internet can be represented by a nonnegative vector [55] gives a necessary and sufficient condition known as the outwardly neighborliness property of the measurement matrix for $\ell_1$ minimization to successfully recover a sparse nonnegative vector. Moreover, recent studies [16, 56, 85] suggested that a sparse solution could be the unique nonnegative solution. This can potentially lead to better alternatives to $\ell_1$-minimization as in this case any optimization problem (with any objective function, for example, $\ell_2$ norm) over this constraint set can recover the original unknown. For instance, although the least squares method with minimizes the $\ell_2$-norm does not promote sparse solutions in general, Slawski and Hein [116] showed when recovering nonnegative sparse signals from low-dimensional noisy measurements, the recovery performance of the nonnegative least squares method is comparable to that of the widely used Lasso method [123]. The message-passing algorithm in [91] for recovering nonnegative sparse signals has time complexity linear in $n$, much faster than $\ell_1$-minimization, but can achieve similar recovery performance as $\ell_1$-minimization. In addition, the sparsest solution can be viewed as a *biased* solution to an underdetermined system, which is undesired in the unbiased networks diagnosis [146]. However, if the uniqueness property holds, the sparse solution is indeed the only nonnegative solution, and thus, unbiased. Therefore, the uniqueness property could be useful in providing unbiased networks diagnosis.

Motivated by networking inference problems such as network tomography, we are particularly interested in systems where the measurement matrix is a 0-1 matrix. There have not been many existing results on this type of systems except a few very recent papers [8, 9, 85, 139]. We focus on two types of binary matrices, Bernoulli 0-1 matrices and adjacency matrices of expanders, and provide conditions under which a sparse vector is the unique nonnegative solution to the underdetermined system. For random Bernoulli measurement matrices, we prove that, as long as the number of equations divided by the number of variables remains constant as the problem dimension grows, with overwhelming probability over the choices of matrices, a sparse nonnegative vector is a unique nonnegative solution provided that its support size is at most proportional to its dimension for some positive ratio. For general expander matrices, we further provide a closed-form constant ratio of support size to dimension under which a nonnegative vector is the unique solution.

The phenomenon that an underdetermined system admits a unique "nonnegative" solution is not restricted to the vector case. Finding the matrix with the minimum rank among all matrices satisfying given linear equations is a *rank minimization* problem. Under this framework, one particularly important class is the rank minimization problem for positive semidefinite matrices under compressed observations. For example, minimizing the rank of a covariance matrix, which is a positive semidefinite matrix, arises in statistics, econometrics, signal processing and many other fields where second-order statistics for random processes are used [64]. A positive semidefinite matrix is special in that its eigenvalues (also its singular values) are nonnegative. In fact, the nuclear norm minimization heuristic for general matrices was preceded by the trace norm heuristic for positive symmetric matrices in rank minimization problems. While the general analytic frameworks and computational techniques, for example, [108, 109], are applicable to the rank minimization problems for positive semidefinite matrices, the special properties of positive semidefinite matrices may open the way to new structures and new analysis, which more efficient computational techniques may exploit to provide faster matrix recovery. Parallel to the influence of the nonnegative constraint on a vector variable, the positive semidefinite constraint on a matrix variable may dramatically reduce the size of the feasible set in rank minimization problems. Interested readers could refer to [131] for this connection.

## 4.2 Unique Nonnegative Vector to an Underdetermined System

If the sparse signal $\mathbf{x}$ to recover is known to be nonnegative, (1.4) is reduced to

$$\min \mathbf{1}^T\mathbf{x} \qquad \text{s.t. } A\mathbf{x} = \mathbf{y}, \mathbf{x} \geq \mathbf{0}. \tag{4.1}$$

In fact, for a certain class of matrices, if $\mathbf{x}$ is sufficiently sparse, not only can we recover $\mathbf{x}$ from (4.1), but also $\mathbf{x}$ is the *only* solution to $\{\mathbf{x} \mid A\mathbf{x} = \mathbf{y}, \mathbf{x} \geq \mathbf{0}\}$. In other words, $\{\mathbf{x} \mid A\mathbf{x} = \mathbf{y}, \mathbf{x} \geq \mathbf{0}\}$ is a singleton. Then $\mathbf{x}$ can possibly be recovered by other techniques to be developed besides $\ell_1$-minimization, since in this case the set $\{\mathbf{x} \mid A\mathbf{x} = \mathbf{y}, \mathbf{x} \geq \mathbf{0}\}$ contains only one solution, which can be recovered by optimizing any objective function over this constraint set.

Bruckstein *et al.* [16] analyzed the singleton property of matrices with a row-span intersecting the positive orthant. Here we first show only these matrices can possibly have the singleton property.

**Definition 4** ([16])**.** *A has a row-span intersecting the positive orthant, denoted by $A \in \mathbf{M}^+$, if $\exists \boldsymbol{\beta} > \mathbf{0}$ ($\boldsymbol{\beta} \in \mathcal{R}^n$) in the row space of A, i.e. $\exists \mathbf{h} \in \mathcal{R}^m$ such that $\mathbf{h}^T A = \boldsymbol{\beta}^T > \mathbf{0}$.*

There is a simple observation regarding matrices in $\mathbf{M}^+$.

**Lemma 11.** *Let $\mathbf{a}_i \in \mathbb{R}^m$ $(i = 1, 2, ..., n)$ be the $i^{th}$ column of matrix A, then $A \in \mathbf{M}^+$ if and only if $\mathbf{0} \notin P$, where*

$$P \triangleq \boldsymbol{Conv}(\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n) = \{A\boldsymbol{\lambda} \mid \mathbf{1}^T\boldsymbol{\lambda} = 1, \boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\lambda} \in \mathcal{R}^n\} \tag{4.2}$$

*Proof.* If $A \in \mathbf{M}^+$, then $\exists \mathbf{h} \in \mathcal{R}^m$ such that $\mathbf{h}^T A = \boldsymbol{\beta}^T > \mathbf{0}$. Suppose we also have $\mathbf{0} \in P$, then $\exists \boldsymbol{\lambda} \geq \mathbf{0} (\boldsymbol{\lambda} \in \mathcal{R}^n)$ such that $A\boldsymbol{\lambda} = \mathbf{0}$ and $\mathbf{1}^T\boldsymbol{\lambda} = 1$. Then $(\mathbf{h}^T A)\boldsymbol{\lambda} = \boldsymbol{\beta}^T\boldsymbol{\lambda} > 0$ as $\boldsymbol{\beta} > \mathbf{0}$, $\boldsymbol{\lambda} \geq \mathbf{0}$ and $\boldsymbol{\lambda} \neq \mathbf{0}$. But $(\mathbf{h}^T A)\boldsymbol{\lambda} = \mathbf{h}^T(A\boldsymbol{\lambda}) = 0$ as $A\boldsymbol{\lambda} = \mathbf{0}$. Contradiction! Therefore $\mathbf{0} \notin P$.

Conversely, if $\mathbf{0} \notin P$, there exists a separating hyperplane $\{\mathbf{x} \mid \mathbf{h}^T\mathbf{x} + b = 0, \mathbf{h} \neq \mathbf{0}\}$ that strictly separates $\mathbf{0}$ and $P$. We assume without loss of generality that $\mathbf{h}^T\mathbf{0} + b < 0$ and $\mathbf{h}^T\mathbf{x} + b > 0$ for any point $\mathbf{x}$ in $P$. Then $\mathbf{h}^T\mathbf{a}_i > -b > 0, \forall i$. Thus we conclude $\mathbf{h}^T A > \mathbf{0}$. ◻

The next result states a necessary condition on matrix $A$ for $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ to be a singleton.

**Proposition 3.** *If* $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ *is a singleton for some* $\mathbf{x}_0 \geq \mathbf{0}$*, then* $A \in \mathbf{M}^+$.

*Proof.* Suppose $A \notin \mathbf{M}^+$, from Lemma 11 we know $\mathbf{0} \in \mathbf{Conv}(a_1, a_2, ..., a_n)$. Then $\exists \mathbf{w} \geq \mathbf{0}$ ($\mathbf{w} \in \mathcal{R}^n$) such that $A\mathbf{w} = \mathbf{0}$ and $\mathbf{1}^T\mathbf{w} = 1$. Clearly $\mathbf{w} \in \mathbf{Null}(A)$ and $\mathbf{w} \neq \mathbf{0}$. Then for any $\gamma > 0$ we have $A(\mathbf{x}_0 + \gamma\mathbf{w}) = A\mathbf{x}_0 + \gamma A\mathbf{w} = A\mathbf{x}_0$, and $\mathbf{x}_0 + \gamma\mathbf{w} \geq \mathbf{0}$ provided $\mathbf{x}_0 \geq \mathbf{0}$. Hence $\mathbf{x}_0 + \gamma\mathbf{w} \in \{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$, and $\mathbf{x}_0 + \gamma\mathbf{w} \neq \mathbf{x}_0$. □

Proposition 3 shows that $A \in \mathbf{M}^+$ is a necessary condition for an underdetermined system to admit a unique nonnegative vector. If $A^{m\times n}$ is a random matrix such that every entry is independently sampled from Gaussian distribution with zero mean, then the probability that $\mathbf{0} \in \mathbf{Conv}(a_1, a_2, ..., a_n)$, or equivalently $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ is not a singleton for any $\mathbf{x}_0 \geq \mathbf{0}$, is $1 - 2^{-n+1}\sum_{k=0}^{m-1}\binom{n-1}{k}$([133]), which goes to 1 asymptotically as $n$ increases if $\lim_{n\to+\infty}\frac{m}{n} < \frac{1}{2}$. Thus, if $\lim_{n\to+\infty}\frac{m}{n} < \frac{1}{2}$, then for a random Gaussian matrix $A$, $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ would not be a singleton with overwhelming probability no matter how sparse $\mathbf{x}_0$ is. This phenomenon is also characterized in [56].

The property that $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ is a singleton can also be characterized in both high-dimensional geometry [56] and the null space property of $A$ [85]. We state three equivalent statements in Theorem 21.

**Theorem 21** ([56][85])**.** *The following three properties of* $A^{m\times n}$ *are equivalent:*

- *For any nonnegative vector* $\mathbf{x}_0 \in \mathcal{R}^n$ *with a support size no greater than* $k$, $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ *is a singleton.*
- *The polytope* $P$ *in (4.2) has* $n$ *vertices and is* $k$*-neighborly.*

- *For any* $\mathbf{w} \neq \mathbf{0}$ ($\mathbf{w} \in \mathcal{R}^n$) *in the null space of A, both the positive support and the negative support of* $\mathbf{w}$ *have a size of at least* $k+1$.

Note that a polytope $P$ is $k$-neighborly if every set of $k$ vertices spans a face $F$ of $P$. $F$ is a face of $P$ if there exists $\boldsymbol{\alpha}_F \in \mathcal{R}^n$ and a constant $c$ such that $\boldsymbol{\alpha}_F^T \mathbf{x} = c, \forall \mathbf{x} \in F$, and $\boldsymbol{\alpha}_F^T \mathbf{x} < c, \forall \mathbf{x} \notin F$ and $\mathbf{x} \in P$.

Donoho and Tanner [56] (Corollary 4.1) showed that there exists a special partial Fourier matrix $\Omega$ with $2p+1$ rows such that $\{\mathbf{x} \mid \Omega\mathbf{x} = \Omega\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ is a singleton for every nonnegative $p$-sparse signal $\mathbf{x}_0$. Here we will show the result is the "best" we can hope for in the sense that a matrix $A$ should have at least $2p+1$ rows if $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ is a singleton for every nonnegative $p$-sparse signal $\mathbf{x}_0$.

**Proposition 4.** *For a matrix* $A^{m \times n}$ $(m < n)$, *if* $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ *is a singleton for any nonnegative p-sparse signal* $\mathbf{x}_0$, *then* $m \geq 2p+1$.

*Proof.* Pick the first $m+1$ columns of $A$, denoted by $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_{m+1} \in \mathbb{R}^m$. Since there are $m$ equations and $m+1$ variables $u_1, u_2, ..., u_{m+1}$ in (4.3), then (4.3) admits a non-zero solution.

$$\sum_{i=1}^{m+1} u_i \mathbf{a}_i = \mathbf{0}. \tag{4.3}$$

From Theorem 3 we know that $A \in \mathbf{M}^+$, i.e. there exists $\mathbf{h}$ such that $\mathbf{h}^T A = \boldsymbol{\beta}^T > \mathbf{0}$. Taking the inner product of both sides of (4.3) with $\mathbf{h}$, we have $\sum_{i=1}^{m+1} \beta_i u_i = 0$.

Since $\boldsymbol{\beta} > \mathbf{0}$, from $\sum_{i=1}^{m+1} \beta_i u_i = 0$ we know vector $\mathbf{u} = (u_i, i = 1, ..., m+1)$ should have both positive and negative terms. Collecting positive and negative terms of $\mathbf{u}$ separatively, we can rewrite (4.3) as follows,

$$\sum_{i\in I_+} u_i\mathbf{a}_i = -\sum_{i\in I_-} u_i\mathbf{a}_i, \tag{4.4}$$

where $I_+$ is the set of indices of positive terms of $\mathbf{u}$ and $I_-$ is the set of indices of negative terms. Note that $|I_+| + |I_-| \leq m+1$. We also have $\sum_{i\in I_+}\beta_i u_i = -\sum_{i\in I_-}\beta_i u_i \triangleq r > 0$ by multiplying $\mathbf{h}^T$ to the left of both sides of (4.4).

Suppose $m \leq 2p$, then $|I_+| + |I_-| \leq m + 1 \leq 2p + 1$, thus we know that $|I_+| \leq p$, or $|I_-| \leq p$, or both hold. Let us first consider the case that $|I_+| \leq p$. Define $B^{n\times n} = \mathrm{diag}(\boldsymbol{\beta})$ and let $D^{m\times n} = AB^{-1}$. Then there is a one-to-one correspondence $\mathbf{z} = B\mathbf{x} \in \mathcal{R}^n$ between the two sets $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ and $\{\mathbf{z} \mid D\mathbf{z} = D\mathbf{z}_0, \mathbf{z} \geq \mathbf{0}\}$, where $\mathbf{z}_0 = B\mathbf{x}_0 \in \mathcal{R}^n$. Note that for any nonnegative and $k$ sparse vector $\mathbf{x}$, $\mathbf{z} = B\mathbf{x}$ is also nonnegative and $k$ sparse. And the converse statement also holds. Since $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ is a singleton for every nonnegative $p$-sparse signal $\mathbf{x}_0$, then $\{\mathbf{z} \mid D\mathbf{z} = D\mathbf{z}_0, \mathbf{z} \geq \mathbf{0}\}$ is also a singleton for every nonnegative $p$-sparse signal $\mathbf{z}_0$. From Theorem 21 $\mathbf{Conv}(\frac{a_1}{\beta_1}, \frac{a_2}{\beta_2}, ..., \frac{a_n}{\beta_n})$ is $p$-neighborly, which implies that for any index set $I$ with $|I| = p$, there exists $\boldsymbol{\eta} \in \mathcal{R}^m$ and constant $c$ such that $\boldsymbol{\eta}^T a_i = \beta_i c$ for any $i \in I$, and $\boldsymbol{\eta}^T a_i < \beta_i c$ for all $i \notin I$. We consider specifically an index set $I$, which contains $I_+$ but does not contain $I_-$, and its corresponding vector $\boldsymbol{\eta}$. Taking the inner product of both sides of (4.4) with $\boldsymbol{\eta}$, we would get $rc$ on the left and some value strictly smaller than $rc$ on the right, and reach a contradiction. For the case that $|I_-| \leq p$ we can reach a contradiction through similar arguments, thus $m \geq 2p + 1$ holds. □

Sparse recovery problems appear in different fields. Specific problem setup may impose further constraints on the measurement matrix. We are particularly interested in network inference problems, in which the measurement matrix is a 0-1 routing matrix. Network inference problems attempt to extract individual parameters based on aggregate measurements in networks. There has been active research in this area including a wide spectrum of approaches ranging from theoretical reasoning to empirical measurements, e.g., [35, 60, 102, 103, 145].

Since the measurement matrices in network inference problems are 0-1 matrices, the instances when $A$ is a 0-1 matrix are our main focus. Section 4.2.1 and 4.2.2 prove that a sparse vector can be the unique nonnegative vector satisfying compressed linear measurements if the measurement matrix is a random Bernoulli matrix or an adjacency matrix of an expander graph. Moreover, the support size of the sparse vector can be proportional to the dimension, in other words, the support size of the unique nonnegative vector is $O(n)$ where $n$ is the dimension, while the provable support size for uniqueness property in [16] is $O(\sqrt{n})$. Besides, for any $\theta \triangleq \lim_{n\to+\infty} \frac{m}{n} > 0$, the support size of a sparse vector that is a unique nonnegative solution can always be $O(n)$, while for Gaussian measurement matrices, with high probability, $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ would not be a singleton for any nonnegative $\mathbf{x}_0$ (with linearly growing sparsity) if $\theta < \frac{1}{2}$ [56]. This also shows the fundamental difference between 0-1 measurement matrices and Gaussian measurement matrices.

### 4.2.1 Uniqueness with 0-1 Bernoulli Matrices

First we consider the uniqueness property with dense 0-1 Bernoulli matrix. The measurement matrix $A$ is an $(m + 1) \times n$ measurement matrix, with each element in the first $m$ rows of $A$ being i.i.d. Bernoulli random variables, taking values '0' with probability $\frac{1}{2}$ and taking values '1' with probability $\frac{1}{2}$. The last row of $A$ is a $1 \times n$ all '1' vector. The fraction ratio $\frac{m}{n}$ is assumed to be a constant $\theta$ as the dimension $n$ grows. It turns out that as $n$ goes to infinity, with overwhelming probability there exists a constant $\gamma > 0$ such that $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ is a singleton for any nonnegative $(\gamma n - 1)$-sparse signal $\mathbf{x}_0$. To see this, we first present the following theorem:

**Theorem 22.** *For any $\theta > 0$, there exists a constant $\gamma > 0$ such that, with overwhelmingly high probability as $n \to \infty$, any nonzero vector $\mathbf{w}$ in the null space of $A$ mentioned above has at least $\gamma n$ negative and at least $\gamma n$ positive elements.*

*Proof.* Let us consider an arbitrary nonzero vector $\mathbf{w} \in \mathcal{R}^n$ in the null space of $A$. Let $S$ be the support set for the negative elements of $\mathbf{w}$ and let $S^c$ be the support set for the nonnegative elements of $\mathbf{w}$. We now want to argue that, with overwhelmingly high probability, the cardinality $|S|$ of the set $S$ can not be too small.

From the large deviation principle and a simple union bound, for any $\epsilon > 0$, with overwhelmingly high probability as $n$ goes to infinity, *simultaneously* for *every* column of the measurement matrix, the sum of its $(m+1)$ elements will be in the range $[\frac{1}{2}\theta(1-\epsilon)n, \frac{1}{2}\theta(1+\epsilon)n]$.

Since $A\mathbf{w} = \mathbf{0}$, then $A_S\mathbf{w}_S + A_{S^c}\mathbf{w}_{S^c} = \mathbf{0}$, where $A_S$, $\mathbf{w}_S$, $A_{S^c}$, and $\mathbf{w}_{S^c}$ are respectively the part of matrix $A$ and vector $\mathbf{w}$ indexed by the sets $S$ and $S^c$. Multiplying an all '1' vector $\mathbf{1}^T$ to both sides of this equation, we get

$$U_S\mathbf{w}_S + U_{S^c}\mathbf{w}_{S^c} = 0, \tag{4.5}$$

where $U_S = \mathbf{1}^T A_S$, $U_{S^c} = \mathbf{1}^T A_{S^c}$ and $\mathbf{1} \in \mathcal{R}^{m+1}$.

From the concentration result of the column sums, we know $U_S\mathbf{w}_S \geq -\frac{1}{2}\theta(1+\epsilon)n\|\mathbf{w}_S\|_1$, and $U_{S^c}\mathbf{w}_{S^c} \geq \frac{1}{2}\theta(1-\epsilon)n\|\mathbf{w}_{S^c}\|_1$. Combining these two inequalities with

(4.5), we have $\frac{1}{2}\theta(1-\epsilon)n\|\mathbf{w}_{S^c}\|_1 - \frac{1}{2}\theta(1+\epsilon)n\|\mathbf{w}_S\|_1 \le 0$, thus,

$$\|\mathbf{w}_S\|_1/\|\mathbf{w}_{S^c}\|_1 \ge (1-\epsilon)/(1+\epsilon). \tag{4.6}$$

Now we look at the null space of the measurement matrix $A$. First, notice that the null space of $A$ is a subset of the null space of the matrix $A'$ comprising of the first $\theta n$ rows of $A$ subtracted by the last row of $A$ (the all '1' vector). Then the matrix $A'$ is a random $\pm 1$ Bernoulli measurement matrix, which is known to satisfy the restricted isometry condition. Recall one result about the null space property of a matrix satisfying the restricted isometry condition:

**Lemma 12** ([24]). *Let $\mathbf{h} \in \mathcal{R}^n$ be any vector in the null space of $A'$ and let $T_0$ be any set of cardinality $q$. Then*

$$\|\mathbf{h}_{T_0}\|_1 \le \sqrt{2}\delta_{2q}\|\mathbf{h}_{T_0^c}\|_1/(1-\delta_{2q}),$$

*where $\delta_{2q}$ is the restricted isometry constant([21]) such that for any set $T$ with $|T| \le 2q$, and any vector $\mathbf{y} \in \mathcal{R}^n$, the following holds:*

$$\sqrt{m}(1-\delta_{2q})\|\mathbf{y}\|_2 \le \|A'_T\mathbf{y}\|_2 \le \sqrt{m}(1+\delta_{2q})\|\mathbf{y}\|_2.$$

Reasoning from Lemma 12 and (4.6), after some algebra, we know immediately, for $q = |S|$, $\delta_{2q}$ must satisfy

$$\delta_{2q} \ge (1-\epsilon)/(1-\epsilon+\sqrt{2}(1+\epsilon)).$$

We also know there exists a $\gamma > 0$ such that for any $q \le \gamma n$, with overwhelmingly high probability as $n \to \infty$,

$$\delta_{2q} < (1-\epsilon)/(1-\epsilon+\sqrt{2}(1+\epsilon)),$$

thus with overwhelmingly high probability as $n \to \infty$, the size of the negative support, namely $|S|$, is at least $\gamma n$.

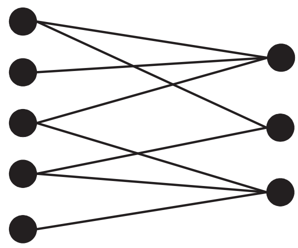


Figure 4.1: The bipartite graph corresponding to matrix $A$ in (4.7)

Similarly, we have the same conclusion for the cardinality of the support set of the positive elements for any nonzero vector from the null space of the matrix $A$. □

Theorem 22 immediately indicates that $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ is a singleton for all nonnegative $\mathbf{x}_0$ that is $\gamma n - 1$ sparse. Thus the support size of the unique nonnegative vector can be as large as $O(n)$, while the previous result in [16] is $O(\sqrt{n})$.

### 4.2.2 Uniqueness with Expander Adjacency Matrices

Section 4.2.1 considers 0-1 Bernoulli matrices, here we consider another type of 0-1 matrices where $A$ is the adjacency matrix of a bipartite expander graph. [9, 85, 139] studied related problems using expander graph with constant left degree. We employ a general definition of expander which does not require constant left degree.

Every $m \times n$ binary matrix $A$ is the adjacency matrix of an unbalanced bipartite graph with $n$ left nodes and $m$ right nodes. There is an edge between right node $i$ and left node $j$ if and only if $A_{ij} = 1$. Let $d_j$ denote the degree of left node $j$, and let $d_l$ and $d_u$ be the minimum and maximum of left degrees. Define $\rho = d_l/d_u$, then $0 < \rho \leq 1$. For example, the bipartite graph in Fig. 4.1 corresponds to the matrix $A$ in (4.7). Here $d_l = 1$, $d_u = 2$, and $\rho = 0.5$.

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}. \tag{4.7}$$

**Definition 5** ([92]). *A bipartite graph with n left nodes and m right nodes is an $(\alpha, \delta)$ expander if for any set $S$ of left nodes of size at most $\alpha n$, $|\Gamma(S)| \geq \delta |E(S)|$ holds, where $E(S)$ is the set of edges connected to nodes in $S$, and $\Gamma(S)$ is the set of right nodes connected to $S$.*

Our next main result regarding the singleton property of an adjacency matrix of a general expander is stated as follows.

**Theorem 23.** *For an adjacency matrix $A$ of an $(\alpha, \delta)$ expander with left degrees in the range $[d_l, d_u]$, if $\delta\rho > \frac{\sqrt{5}-1}{2} \approx 0.618$, then for any nonnegative k-sparse vector $\mathbf{x}_0$ with $k \leq \frac{\alpha}{1+\delta\rho} n$, $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ is a singleton.*

*Proof.* From Theorem 21, to prove that $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ is a singleton for any nonnegative $\frac{\alpha n}{1+\delta\rho}$-sparse vector $\mathbf{x}_0$, we only need to argue that for any nonzero $\mathbf{w} \in \mathbf{Null}(A)$ with $S_-$ and $S_+$ as its negative support and positive support, $|S_-| \geq \frac{\alpha n}{1+\delta\rho} + 1$ and $|S_+| \geq \frac{\alpha n}{1+\delta\rho} + 1$ hold.

We will prove by contradiction. Suppose without loss of generality that there exists a nonzero $\mathbf{w} \in \mathbf{Null}(A)$ such that $|S_-| = s \leq \frac{\alpha n}{1+\delta\rho}$, then the set $E(S_-)$ of edges connected to nodes in $S_-$ satisfies $d_l s \leq |E(S_-)| \leq d_u s$. Then the set $\Gamma(S_-)$ of neighbors of $S_-$ satisfies

$$d_u s \geq |E(S_-)| \geq |\Gamma(S_-)| \geq \delta |E(S_-)| \geq \delta d_l s,$$

where the third equality comes from the expander property.

Notice that $\Gamma(S_-) = \Gamma(S_+) = \Gamma(S_- \cup S_+)$, since otherwise $Aw = \mathbf{0}$ does not hold, then $|S_+| \geq |\Gamma(S_+)|/d_u = |\Gamma(S_-)|/d_u \geq \delta d_l s/d_u = \delta\rho s$.

Now consider the set $S_- \cup S_+$, we have $|S_- \cup S_+| \geq (1+\delta\rho)s$. Pick an arbitrary subset $\tilde{S} \in S_- \cup S_+$ such that $|\tilde{S}| = (1+\delta\rho)s \leq \alpha n$. From expander property, we have

$$|\Gamma(\tilde{S})| \geq \delta|E(\tilde{S})| \geq \delta d_l|\tilde{S}| = \delta\rho(1+\delta\rho)d_u s > d_u s.$$

The last inequality holds since $\delta\rho(1+\delta\rho) > 1$ provided $\delta\rho > \frac{\sqrt{5}-1}{2}$. But $|\Gamma(\tilde{S})| \leq |\Gamma(S_- \cup S_+)| = |\Gamma(S_-)| \leq d_u s$. A contradiction arises, which completes the proof. $\square$

**Corollary 4.** *For an adjacency matrix $A$ of an $(\alpha, \delta)$ expander with constant left degree $d$, if $\delta > \frac{\sqrt{5}-1}{2}$, then for any nonnegative $k$-sparse vector $\mathbf{x}_0$ with $k \leq \frac{\alpha}{1+\delta}n$, $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ is a singleton.*

Theorem 23 together with Corollary 4 is an extension to existing results. Theorem 3.5 of [85] shows that for an $(\alpha, \delta)$ expander with constant left degree $d$, if $d\delta > 1$, then there exists a matrix $\tilde{A}$ (a perturbation of $A$) such that $\{\mathbf{x} \mid \tilde{A}\mathbf{x} = \tilde{A}\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ is a singleton for every nonnegative $\delta\alpha n$-sparse $\mathbf{x}_0$. Our result instead can directly quantify the sparsity threshold needed for a vector to be a unique solution to compressed measurements induced by $A$, not its perturbation. [9] discussed the success of $\ell_1$ recovery of a general vector $\mathbf{x}$ for expanders with constant left degree. If we apply Theorem 1 of [9] to cases where $\mathbf{x}$ is known to be nonnegative, the result can be interpreted as that $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ is a singleton for any nonnegative $\frac{\alpha}{2}n$-sparse vector $\mathbf{x}_0$ if $\delta > \frac{5}{6} \approx 0.833$. Our result in Corollary 4 implies that if $\delta > \frac{\sqrt{5}-1}{2} \approx 0.618$, $\mathbf{x}_0$ can be $\frac{\alpha}{1+\delta}n$-sparse and still be the unique nonnegative solution.

Sipser and Spielman [115], and Feldman *et al.* [65] proved that for any $m$, $n$ and $\delta > 0$, there exists an $(\alpha, \delta)$ expander with constant left degree $d$ for some $d$ and $\alpha > 0$, and such an expander can be generated through random graphs. There also exist explicit constructions of expander graphs [25]. Combining the results with Corollary 4, for any $m$ and $n$, we can generate an $(\alpha, \delta)$ expander with adjacency matrix $A$ such that $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ is a singleton for any nonnegative $kn$-sparse $\mathbf{x}_0$, where $k = \frac{\alpha}{1+\delta} > 0$. Thus, same as Bernoulli 0-1 matrices, the adjacency matrix $A$ of an $(\alpha, \delta)$ expander has the property that $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ is a singleton as long as the support size of $\mathbf{x}_0$ is $O(n)$. We further provide an explicit constant $\frac{\alpha}{1+\delta}$ of the ratio of the support size to the dimension. Note that this result is independent of $\frac{m}{n}$, while as discussed earlier, if the matrix has i.i.d. Gaussian entries and $\lim_{n\to+\infty} \frac{m}{n} < \frac{1}{2}$, $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ is not a singleton despite the sparsity of $\mathbf{x}_0$.

## 4.3 Simulation

We generate a random 0-1 matrix $A^{m\times n}$ with i.i.d. entries and empirically study the uniqueness property and the success of $\ell_1$ minimization for nonnegative vectors with different sparsity. Each entry of $A$ takes value 1 with probability 0.2 and value 0 with probability 0.8. The size of $A$ is $50 \times 200$ and $100 \times 200$ respectively. For a sparsity $k$, we select a support set $S$ with size $|S| = k$ uniformly at random, and generate a nonnegative vector $\mathbf{x}_0$ on $S$ with i.i.d. entries uniformly on the unit interval. Then we check whether $U \triangleq \{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ is singleton. This can be realized as follows. We minimize and maximize the same objective function $\mathbf{d}^T\mathbf{x}$ over $U$, where $\mathbf{d}$ is a random vector in $\mathbb{R}^n$. Note that if $U$ is not a singleton, then the set $\{\mathbf{d} \in \mathbb{R}^n \mid \mathbf{d}^T\mathbf{x} = \mathbf{d}^T\mathbf{x}_0, \forall \mathbf{x} \in U\}$ has measure 0.

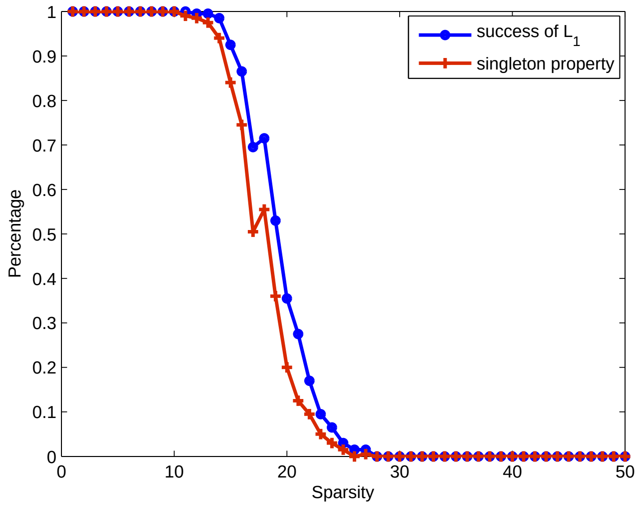



Figure 4.2: Comparison of $\ell_1$ recovery and singleton property for a $50 \times 200$ 0-1 matrix

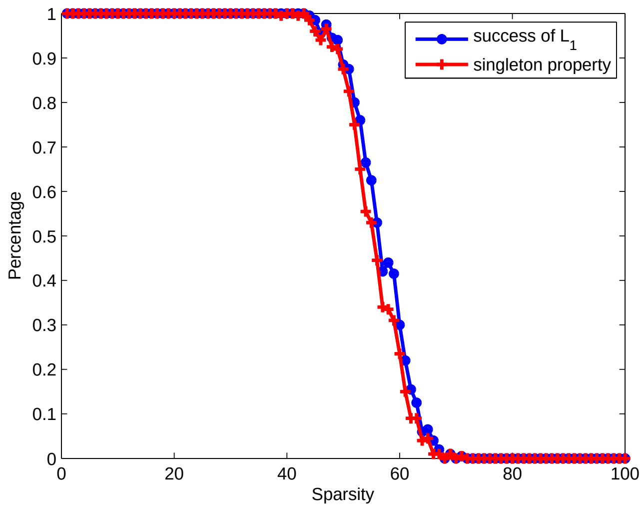



Figure 4.3: Comparison of $\ell_1$ recovery and singleton property for a $100 \times 200$ 0-1 matrix

Thus the probability that the minimizer and the maximizer are the same when $U$ is not a singleton is 0. We generate several different $\mathbf{d}$'s and claim $U$ to be singleton if the minimizer and the maximizer are the same for every $\mathbf{d}$. For each instance, we also check whether $\ell_1$ minimization can recover $\mathbf{x}_0$ from $A\mathbf{x}_0$ or not. Under a given sparsity $k$, we generate 200 $\mathbf{x}_0$'s and repeat the above procedure 200 times.

We fix $n$ to be 200, and $m$ is 50 in Fig. 4.2 and 100 in Fig. 4.3 . When $m/n$ increases from 1/4 to 1/2, the support size of a sparse vector which is a unique nonnegative solution increases from $0.05n$ to $0.19n$. Note that when $m/n = 1/2$, for this 0-1 matrix, the singleton property still exists linearly in $n$, while for a random Gaussian matrix, with overwhelming probability no vector can be a unique nonnegative solution. Besides, the thresholds where the singleton property breaks down and where the fully recovery of $\ell_1$ minimization breaks down are quite close.

CHAPTER 5

# CONCLUSION AND FUTURE WORK

Sparse recovery provides the opportunity to represent sparse signals with an incomplete set of indirect observations. Developing efficient sparse recovery algorithms with performance guarantee and designing measurement matrices suitable for sparse recovery are two important questions to address.

**Theoretical analysis of sparse recovery performance.**

In Chapter 2, we analyzed a class of sparse recovery methods known as $\ell_p$-minimization ($0 \leq p < 1$), characterized the theoretical performance guarantee and compared with the widely-used $\ell_1$-minimization method. When $\alpha = m/n \to 1$, we provided a tight threshold $\rho^*(p)$ of the sparsity ratio separating the success and failure of strong recovery which requires to recover all the sparse vectors. $\rho^*(p)$ strictly decreases from 0.5 to 0.239 as $p$ increases from 0 to 1. For weak recovery which only needs to recover sparse vectors on some support with some sign pattern, we first provided an equivalent null space characterization of successful weak recovery, then proved that the threshold of sparsity ratio separating the success and failure of $\ell_p$-minimization is $2/3$ for all $p < 1$, compared with the threshold 1 for $\ell_1$-minimization. We also explicitly demonstrated that $\ell_p$-minimization ($p < 1$) can return a denser solution than $\ell_1$-minimization. For any $\alpha < 1$, we provided a bound $\rho^*(\alpha, p)$ of sparsity ratio below which strong recovery via $\ell_p$-minimization succeeds with overwhelming probability, and our bound $\rho^*(\alpha, p)$ improves on the existing bounds in the large $\alpha$ region. We also provided a bound $\rho^*_w(\alpha, p)$ of sparsity ratio below which weak recovery succeeds with overwhelming probability.

We argued that $\ell_p$-minimization has a higher threshold with smaller $p$ for strong recovery; the threshold is the same for all $p$ for sectional recovery; and $\ell_1$-minimization can outperform $\ell_p$-minimization for weak recovery. These are in contrast to traditional wisdom that $\ell_p$-minimization, though computationally more expensive, always has better sparse recovery ability than $\ell_1$-minimization since it is closer to $\ell_0$-minimization.

Throughout the analysis, we assumed that the measurements $\mathbf{y} = A\mathbf{x}$ are exact, and it would be interesting to consider the case that the measurements are noisy, i.e. $\mathbf{y} = A\mathbf{x} + \mathbf{e}$ where $\mathbf{e}$ is the vector of noise. Moreover, we assumed that $\mathbf{x}$ is exactly sparse, i.e. most of its entries are exactly zero. The extension of results to approximately sparse vectors whose coefficients (if ordered) decay rapidly requires further efforts.

**Measurement construction with graph constraints.**

Sparse recovery has drawn much attention in the signal processing and general systems community in the last couple of years, but its applications in networks are still limited. The application of sparse recovery in network monitoring problems bring up new directions to explore.

In Chapter 3, we discussed constructions of sparse recovery measurements in the presence of additional graph topological constraints that have not been addressed in the literature of sparse recovery. We provided explicit measurement constructions for special graphs and proposed measurement construction algorithm for general graphs. Our construction for a line network is provably optimal in the sense that it requires the minimum number of measurements. We also characterized the relation between the number of measurements needed for sparse recovery and the graph structure. With additional graph topological constraints, only nodes that satisfy certain type of graph connectivity can be aggregated in one measurement. Provided that the connectivity of a graph is not poor, the number of measurements needed for sparse recovery by our construction methods is still close to that needed in the conventional one without graph constraints. We also derive upper and lower bounds of the minimum number of measurements needed for sparse recovery on a given graph. It would be interesting to tighten such bounds, especially the lower bounds. We focus on the topological constraints that one can only take aggregate measurements over nodes that induce a connected subgraph, and it is intriguing to explore other type of topological constraints.

**Measurement construction with graph constraints.**

In Chapter 4, we analyzed the phenomenon that a sparse nonnegative signal is the only nonnegative solution to an underdetermined linear system. It can lead to efficient recovery algorithms since the incomplete set of equations only admits a unique nonnegative solution.

We showed that only for a class of matrices with a row span intersecting the positive orthant that $\{\mathbf{x} \mid A\mathbf{x} = A\mathbf{x}_0, \mathbf{x} \geq \mathbf{0}\}$ could possibly be a singleton if $\mathbf{x}_0$ is sparse enough. Among these matrices, we are interested in 0-1 matrices which fit the setup of network inference problems.

For Bernoulli 0-1 matrices, we proved that with high probability the unique solution property holds for all $k$-sparse nonnegative vectors where $k$ is $O(n)$, instead of the previous result $O(\sqrt{n})$. For the adjacency matrix of a general expander, we proved the existence of the same phenomenon and further provided a closed-form constant ratio of $k$ to $n$.

One future direction is to obtain uniqueness property threshold for a given measurement matrix. So far, we have only discussed the ideally sparse nonnegative vectors, but we would also like to consider recovering approximately sparse nonnegative signal vectors. In approximate sparse recovery problems, instead of being a singleton, the feasible set can contain an infinite number of solutions, but we conjecture its measure is "small".

The following questions related to this dissertation are also worth further exploration.

**Sparse recovery over networks: from theory to practice.**

We discussed measurement constructions with additional topological constraints in Chapter 2. Our constructions are based on a simplified model where a graph captures the topological constraints, and some practical issues have not been considered yet. We assumed the network topology is fixed and known. If the network topology is partially known or changes over time (e.g., in the sensor networks), is it still possible to apply sparse recovery in this case? If the answer is yes, how can we construct sparse recovery measurements with partially known or time-varying topological constraints? Moreover, we only designed linear measurements such that every sparse signal has a distinct low-dimensional representation and did not consider the recovery algorithms. One immediate step is to design sparse recovery algorithms over graphs and analyze the recovery performance, especially when the measurements are noisy or the signals are approximately sparse. Furthermore, our results are theoretical and not directly applicable to network monitoring problems. For example, we assumed that nodes can be aggregated together in a measurement as long as they induce a connected subgraph; further efforts are required to design corresponding network transmission protocols for implementation.

**A compressed representation of large data: existence, compression and reconstruction.**

Sparse recovery indicates that it is possible to characterize a high-dimensional object by a small number of linear projections. Sparsity in the vector space and low rank property in the matrix space are examples of special geometry that allows a compressed low-dimensional representation. Other forms of geometry that can lead to a compressed representation are worth pursuing. In large-scale system, given the massive amount of data, is it possible to identify the existence of any special structure as well as the resulting compressed representation? If so, can we develop efficient compression algorithms and reconstruction algorithms?

APPENDIX A

**PROOFS OF CHAPTER 1**

## A.1 Proof of Theorem 3

*Proof.* Necessary part. Suppose the condition fails for some $\mathbf{z}$, then there are two cases: (1) $T^+$ is empty, and (2) $T^+$ is not empty for that particular $\mathbf{z}$.

First consider the case $T^+$ is empty, then we have $||B_{T^-}\mathbf{z}||_p^p \geq ||B_{T^c}\mathbf{z}||_p^p$ since we assume the condition in Theorem 3 fails for $\mathbf{z}$. Define a vector $\mathbf{x}$ as follows. Let $x_i = 0$ for every $i$ in $T^c$, let $x_i = -B_i\mathbf{z}$ for every $i$ in $T^-$. Let $x_i$ be any positive value for every $i$ in $T^0$. Then according to the definition of $\mathbf{x}$, we have

$$\begin{aligned}
& ||\mathbf{x} + B\mathbf{z}||_p^p \\
= \ & ||\mathbf{x}_{T^-} + B_{T^-}\mathbf{z}||_p^p + ||\mathbf{x}_{T^0} + B_{T^0}\mathbf{z}||_p^p + ||B_{T^c}\mathbf{z}||_p^p \\
= \ & 0 + ||\mathbf{x}_{T^0}||_p^p + ||B_{T^c}\mathbf{z}||_p^p \\
= \ & ||\mathbf{x}||_p^p - ||\mathbf{x}_{T^-}||_p^p + ||B_{T^c}\mathbf{z}||_p^p \\
= \ & ||\mathbf{x}||_p^p - ||B_{T^-}\mathbf{z}||_p^p + ||B_{T^c}\mathbf{z}||_p^p \\
\leq \ & ||\mathbf{x}||_p^p.
\end{aligned}$$

Since $||\mathbf{x} + B\mathbf{z}||_p^p \leq ||\mathbf{x}||_p^p$, (2.1) cannot successfully recover $\mathbf{x}$, which is a contradiction.

Secondly, consider the case that $T^+$ is not empty. Then $||B_{T^-}\mathbf{z}||_p^p > ||B_{T^c}\mathbf{z}||_p^p$ since we assume the condition in Theorem 3 fails for $\mathbf{z}$. Let $\delta = ||B_{T^-}\mathbf{z}||_p^p - ||B_{T^c}\mathbf{z}||_p^p > 0$. Define a vector $\mathbf{x}$ as follows. Let $x_i = 0$ for every $i$ in $T^c$, let $x_i = -B_i\mathbf{z}$ for every $i$ in $T^-$, and let $x_i$ be any positive value for every $i$ in $T^0$. For every $i$ in $T^+$, since $p \in (0,1)$, we can pick $x_i > 0$ large enough such that $||\mathbf{x}_{T^+} + B_{T^+}\mathbf{z}||_p^p - ||\mathbf{x}_{T^+}||_p^p < \frac{\delta}{2}$.

Then

$$\begin{aligned}
\|\mathbf{x} + B\mathbf{z}\|_p^p &= 0 + \|\mathbf{x}_{T^+} + B_{T^+}\mathbf{z}\|_p^p + \|\mathbf{x}_{T^0}\|_p^p + \|B_{T^c}\mathbf{z}\|_p^p \\
&< \|\mathbf{x}_{T^+}\|_p^p + \frac{\delta}{2} + \|\mathbf{x}_{T^0}\|_p^p + \|B_{T^c}\mathbf{z}\|_p^p \\
&= \|\mathbf{x}_{T^+}\|_p^p + \frac{\delta}{2} + \|\mathbf{x}_{T^0}\|_p^p + \|B_{T^-}\mathbf{z}\|_p^p - \delta \\
&= \|\mathbf{x}\|_p^p - \frac{\delta}{2}.
\end{aligned}$$

Thus $\|\mathbf{x} + B\mathbf{z}\|_p^p < \|\mathbf{x}\|_p^p$, $\mathbf{x}$ is not a solution to (2.1), which is also a contradiction.

Sufficient part. Assume the null space condition holds, then for any nonnegative $\mathbf{x}$ on support $T$, and any non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$, we have

$$\begin{aligned}
& \|\mathbf{x} + B\mathbf{z}\|_p^p \\
= \;& \|\mathbf{x}_{T^+} + B_{T^+}\mathbf{z}\|_p^p + \|\mathbf{x}_{T^-} + B_{T^-}\mathbf{z}\|_p^p \\
& + \|\mathbf{x}_{T^0}\|_p^p + \|B_{T^c}\mathbf{z}\|_p^p \\
\geq \;& \|\mathbf{x}_{T^+} + B_{T^+}\mathbf{z}\|_p^p + \|\mathbf{x}_{T^-}\|_p^p - \|B_{T^-}\mathbf{z}\|_p^p \\
& + \|\mathbf{x}_{T^0}\|_p^p + \|B_{T^c}\mathbf{z}\|_p^p, \qquad \text{(A.1)}
\end{aligned}$$

where the inequality follows from the triangular property that $|\mathbf{x}_i + B_i\mathbf{z}|^p \geq |\mathbf{x}_i|^p - |B_i\mathbf{z}|^p$ holds for all $i$ and all $p \in (0, 1)$.

If $T^+$ is not empty, then $\|\mathbf{x}_{T^+} + B_{T^+}\mathbf{z}\|_p^p > \|\mathbf{x}_{T^+}\|_p^p$ since $B_i\mathbf{z} > 0$ for every $i$ in $T^+$, and $B_i\mathbf{z}$ and $x_i$ have the same sign. Since we also have $\|B_{T^-}\mathbf{z}\|_p^p \leq \|B_{T^c}\mathbf{z}\|_p^p$ from assumption, therefore by (A.1) we have $\|\mathbf{x} + B\mathbf{z}\|_p^p > \|\mathbf{x}\|_p^p$. If $T^+$ is empty, then $\|B_{T^-}\mathbf{z}\|_p^p < \|B_{T^c}\mathbf{z}\|_p^p$ from assumption, therefore by (A.1) we also have $\|\mathbf{x} + B\mathbf{z}\|_p^p > \|\mathbf{x}\|_p^p$. Thus, $\|\mathbf{x} + B\mathbf{z}\|_p^p > \|\mathbf{x}\|_p^p$ for all non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$, then $\mathbf{x}$ is the solution to (2.1). □

## A.2 Proof of Lemma 1

*Proof.* Let $X \sim \mathcal{N}(0, 1)$ and let $Z = |X|$. Let $f(z)$ and $F(z)$ denote the p.d.f. and c.d.f. of $Z$ respectively. Then

$$f(z) = \begin{cases} \sqrt{2/\pi}e^{-\frac{1}{2}z^2} & \text{if } z \geq 0, \\ 0 & \text{if } z < 0. \end{cases} \tag{A.2}$$

$$F(z) = \begin{cases} \operatorname{erf}(z/\sqrt{2}) = \int_0^z \sqrt{2/\pi}e^{-\frac{1}{2}x^2}dx, & \text{if } z \geq 0 \\ 0 & \text{if } z < 0. \end{cases} \tag{A.3}$$

Define $g(t) = \int_t^\infty z^p f(z)dz$. $g$ is continuous and decreasing in $[0, \infty]$, and $g(0) = E[Z^p] = \frac{S}{n}$, $\lim_{t\to\infty} g(t) = 0$. Then there exists $z^*$ such that $g(z^*) = \frac{g(0)}{2}$, i.e.

$$\int_0^{z^*} x^p f(x)dx - \int_{z^*}^\infty x^p f(x)dx = 0. \tag{A.4}$$

Define

$$\rho^* = 1 - F(z^*). \tag{A.5}$$

We claim $\rho^*$ has the desired property.

Let

$$T_{z^*} = \sum_{i:Y_i \geq z^{*p}} Y_i = \sum_{i=1}^n Y_i \mathbf{1}_{\{Y_i \geq z^{*p}\}},$$

where $\mathbf{1}$ is the indicator function. Then

$$\begin{aligned} E[T_{z^*}] &= E[\sum_{i=1}^n Y_i \mathbf{1}_{\{Y_i \geq z^{*p}\}}] = E[\sum_{i=1}^n |X_i|^p \mathbf{1}_{\{|X_i|^p \geq z^{*p}\}}] \\ &= nE[Z^p \mathbf{1}_{\{Z \geq z^*\}}] = n\int_{z^*}^\infty z^p f(z)dz = ng(z^*). \end{aligned}$$

Let $h$ be the smallest integer such that $Y_h \geq z^{*p}$ and $Y_{h+1} < z^{*p}$, then $T_{z^*} = \sum_{i=1}^h Y_i$. We also have that $h = \sum_{i=1}^n \mathbf{1}_{\{|X_i|^p \geq z^{*p}\}} = \sum_{i=1}^n \mathbf{1}_{\{|X_i| \geq z^*\}}$. Note that $P(|X_i| \geq z^*) = 1 -$

$F(z^*) = \rho^*$, thus $h$ follows the Binomial distribution $B(n, \rho^*)$. Then its expectation $E[h] = \rho^* n$, and the variance $E[(h - \rho^* n)^2] = n\rho^*(1 - \rho^*)$.

We claim that

$$|T_{z^*} - S_{\rho^*}| \leq \frac{|h - \lceil \rho^* n \rceil| S_{\rho^*}}{\lceil \rho^* n \rceil}. \tag{A.6}$$

To see this, consider three different cases, $h = \lceil \rho^* n \rceil$, $h > \lceil \rho^* n \rceil$ and $h < \lceil \rho^* n \rceil$. If $h = \lceil \rho^* n \rceil$, then $T_{z^*} = S_{\rho^*}$, and (A.6) holds trivially. If $h > \lceil \rho^* n \rceil$, then $|T_{z^*} - S_{\rho^*}| = \sum_{i=\lceil \rho^* n \rceil + 1}^{h} Y_i$. Note that for every $i > \lceil \rho^* n \rceil$, $Y_i \leq Y_{\lceil \rho^* n \rceil} \leq S_{\rho^*}/\lceil \rho^* n \rceil$, then (A.6) follows. If $h < \lceil \rho^* n \rceil$, then $|T_{z^*} - S_{\rho^*}| = \sum_{i=h+1}^{\lceil \rho^* n \rceil} Y_i$. Since $Y_i \geq Y_j$ for all $i \leq j$, then $\sum_{i=h+1}^{\lceil \rho^* n \rceil} Y_i/(\lceil \rho^* n \rceil - h) \leq \sum_{i=1}^{h} Y_i/h$, which leads to $\sum_{i=h+1}^{\lceil \rho^* n \rceil} Y_i/(\lceil \rho^* n \rceil - h) \leq \sum_{i=1}^{\lceil \rho^* n \rceil} Y_i/\lceil \rho^* n \rceil = S_{\rho^*}/\lceil \rho^* n \rceil$, and (A.6) follows. Combining three cases, we conclude that (A.6) always holds. Then

$$\begin{aligned} E[|T_{z^*} - S_{\rho^*}|] &\leq \frac{E[|h - \lceil \rho^* n \rceil| S_{\rho^*}]}{\lceil \rho^* n \rceil} \\ &\leq \frac{\sqrt{E[(h - \lceil \rho^* n \rceil)^2] E[S_{\rho^*}^2]}}{\lceil \rho^* n \rceil}, \end{aligned} \tag{A.7}$$

where the second inequality follows from the Cauchy-Schwarz inequality. We have

$$\begin{aligned} & E[(h - \lceil \rho^* n \rceil)^2] \\ = \; & E[(h - \rho^* n)^2] + 2(\rho^* n - \lceil \rho^* n \rceil) E[h - \rho^* n] \\ & + (\rho^* n - \lceil \rho^* n \rceil)^2 \\ \leq \; & n\rho^*(1 - \rho^*) + 1. \end{aligned}$$

Besides,

$$\begin{aligned} E[S_{\rho^*}^2] &\leq E[S_1^2] = E[(\sum_{i=1}^{n} |X_i|^p)^2] \\ &= E[\sum_{i=1}^{n} |X_i|^{2p} + \sum_{i,j:i \neq j} |X_i|^p |X_j|^p] \\ &= nE[|X|^{2p}] + n(n-1)(E[|X|^p])^2, \end{aligned}$$

where the third equality follows since $X_1, X_2,...,X_n$ are i.i.d. $\mathcal{N}(0,1)$ random variables. Then from (A.7) we have

$$\begin{aligned} & E[|T_{z^*} - S_{\rho^*}|] \\ \leq \;& \frac{\sqrt{(n\rho^*(1-\rho^*)+1)(nE[|X|^{2p}] + n(n-1)(E[|X|^p])^2)}}{\lceil \rho^* n \rceil} \\ = \;& O(\sqrt{n}). \end{aligned}$$

Since $E[|T_{z^*} - S_{\rho^*}|]$ is upper bounded by $O(\sqrt{n})$, $E[T_{z^*}] = ng(z^*)$, and $S = ng(0)$, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{E[S_{\rho^*}]}{S} &= \lim_{n\to\infty} \frac{E[T_{z^*}]}{S} + \lim_{n\to\infty} \frac{E[S_{\rho^*} - T_{z^*}]}{S} \\ &= \frac{g(z^*)}{g(0)} + 0 = \frac{1}{2}. \end{aligned}$$

□

## A.3 Proof of Proposition 1

*Proof.* From the definition of $z^*$ in (A.4), we have

$$H(z^*, p) := \int_0^{z^*} x^p f(x)dx - \int_{z^*}^{\infty} x^p f(x)dx = 0, \tag{A.8}$$

where $f(\cdot)$ and $F(\cdot)$ are defined in (A.2) and (A.3). From the Implicit Function Theorem,

$$\frac{dz^*}{dp} = -\frac{\frac{\partial H}{\partial p}}{\frac{\partial H}{\partial z^*}} = -\frac{\int_0^{z^*} x^p(\ln x) f(x)dx - \int_{z^*}^{\infty} x^p(\ln x) f(x)dx}{2z^{*p} f(z^*)}.$$

From (A.5), we have $\frac{d\rho^*}{dz^*} = -f(z^*)$. From the chain rule, we know $\frac{d\rho^*}{dp} = \frac{d\rho^*}{dz^*}\frac{dz^*}{dp}$, thus

$$\frac{d\rho^*}{dp} = \frac{\int_0^{z^*} x^p(\ln x) f(x)dx - \int_{z^*}^{\infty} x^p(\ln x) f(x)dx}{2z^{*p}} \tag{A.9}$$

Note that

$$\begin{aligned}
\int_0^{z^*} x^p(\ln x) f(x) dx &< \int_0^{z^*} x^p(\ln z^*) f(x) dx \\
&= \int_{z^*}^{\infty} x^p(\ln z^*) f(x) dx \\
&< \int_{z^*}^{\infty} x^p(\ln x) f(x) dx, \qquad \text{(A.10)}
\end{aligned}$$

where the equality follows from (A.8). Then the numerator of (A.9) is less than 0 from (A.10), thus $\frac{d\rho^*}{dp} < 0$. □

## A.4 Proof of Lemma 2

*Proof.* Let $\mathbf{X} = [X_1, ..., X_n]^T$. If two vectors $\mathbf{X}$ and $\mathbf{X}'$ only differ in co-ordinate $i$, then for any $p$, $|S_\rho(\mathbf{X}) - S_\rho(\mathbf{X}')| \leq ||X_i|^p - |X_i'|^p|$. Thus for any $\mathbf{X}$ and $\mathbf{X}'$,

$$|S_\rho(\mathbf{X}) - S_\rho(\mathbf{X}')| \leq \sum_{i:X_i \neq X_i'} \left||X_i|^p - |X_i'|^p\right|.$$

Since $\left||X_i|^p - |X_i'|^p\right| \leq |X_i - X_i'|^p$ for all $p \in (0, 1]$,

$$|S_\rho(\mathbf{X}) - S_\rho(\mathbf{X}')| \leq \sum_i |X_i - X_i'|^p. \qquad \text{(A.11)}$$

From the isoperimetric inequality for the Gaussian measure [88], for any set $A \in \mathcal{R}^n$ with measure at least a half, the set $A_t = \{\mathbf{x} \in \mathcal{R}^n : d(\mathbf{x}, A) \leq t\}$ has measure at least $1 - e^{-t^2/2}$, where $d(\mathbf{x}, A) = \inf_{\mathbf{y} \in A} ||\mathbf{x} - \mathbf{y}||_2$. Let $M_\rho$ be the median value of $S_\rho = S_\rho(\boldsymbol{X})$. Define set $A = \{\mathbf{x} \in \mathcal{R}^n : S_\rho(\mathbf{x}) \leq M_\rho\}$, then

$$P(d(\mathbf{x}, A) \leq t) \geq 1 - e^{-t^2/2}.$$

We claim that $d(\mathbf{x}, A) \leq t$ implies that $S_\rho(\mathbf{x}) \leq M_\rho + n^{(1-p/2)} t^p$. If $\mathbf{x} \in A$, then $S_\rho(\mathbf{x}) \leq M_\rho$, thus the claim holds as $n^{1-p/2} t^p$ is nonnegative. If $\mathbf{x} \notin A$, then there exists $\mathbf{x}' \in A$ such that $||\mathbf{x} - \mathbf{x}'||_2 \leq t$. Let $u_i = 1$ for all $i$ and let $v_i = |x_i - x_i'|^p$. From Hölder's inequality,

$$\begin{aligned}
\sum_i |x_i - x_i'|^p &\leq \left(\sum_i |u_i|^{2/(2-p)}\right)^{1-p/2} \left(\sum_i |v_i|^{2/p}\right)^{p/2} \\
&\leq n^{(1-p/2)}(t^2)^{p/2} = n^{(1-p/2)}t^p \qquad \text{(A.12)}
\end{aligned}$$

From (A.11) and (A.12), $|S_\rho(\mathbf{x}) - S_\rho(\mathbf{x}')| \leq n^{(1-p/2)}t^p$. Since $\mathbf{x} \notin A$ and $\mathbf{x}' \in A$, then $S_\rho(\mathbf{x}) > M_\rho \geq S_\rho(\mathbf{x}')$. Thus $S_\rho(\mathbf{x}) \leq M_\rho + n^{(1-p/2)}t^p$, which verifies our claim. Then

$$P(S_\rho(\mathbf{x}) \leq M_\rho + n^{(1-p/2)}t^p) \geq P(d(\mathbf{x}, A) \leq t) \geq 1 - e^{-t^2/2}. \qquad \text{(A.13)}$$

Similarly,

$$P(S_\rho(\mathbf{x}) \geq M_\rho - n^{(1-p/2)}t^p) \geq 1 - e^{-t^2/2}. \qquad \text{(A.14)}$$

Combining (A.13) and (A.14),

$$P(|S_\rho(x) - M_\rho| \geq n^{(1-p/2)}t^p) \leq 2e^{-t^2/2}. \qquad \text{(A.15)}$$

The difference of $E[S_\rho]$ and $M_\rho$ can be bounded as follows,

$$\begin{aligned}
|E[S_\rho] - M_\rho| &\leq E[|S_\rho - M_\rho|] \\
&= \int_0^\infty P(|S_\rho(x) - M_\rho| \geq y)dy \\
&\leq \int_0^\infty 2e^{-\frac{1}{2}y^{\frac{2}{p}}n^{(1-\frac{2}{p})}}dy \\
&= n^{(1-\frac{p}{2})}\int_0^\infty 2e^{-\frac{1}{2}s^{\frac{2}{p}}}ds
\end{aligned}$$

Note that $c := \int_0^\infty 2e^{-\frac{1}{2}s^{(2/p)}}ds$ is a finite constant for all $p \in (0, 1]$. As $p > 0$ and $S = nE[|x_i|^p]$, thus for any $\delta > 0$, $cn^{(1-\frac{p}{2})} < \frac{\delta}{2}S$ when $n$ is large enough.

Let $t = \left(\frac{1}{2}\delta S n^{(\frac{p}{2}-1)}\right)^{\frac{1}{p}} = (\frac{1}{2}\delta E[|x_i|^p])^{\frac{1}{p}}\sqrt{n}$, from (A.15) with probability at least $1-2e^{-\frac{1}{2}(\frac{1}{2}\delta E[|x_i|^p])^{\frac{2}{p}}n}$, $|S_\rho - M_\rho| < \frac{1}{2}\delta S$. Thus $|S_\rho - E[S_\rho]| \leq |S_\rho - M_\rho| + |M_\rho - E[S_\rho]| < \delta S$ with probability at least $1 - 2e^{-c_1 n}$ for some constant $c_1$. □

## A.5 Proof of Corollary 1

*Proof.* From Lemma 1 we know that for every $\epsilon > 0$, there exists $M$ large enough such that

$$E[S_{\rho^*}] \leq (\frac{1}{2} + \epsilon)S \tag{A.16}$$

for all $n \geq M$ where $S = E[S_1]$. Then $E[\sum_{i=\lceil\rho^* n\rceil+1}^{n} Y_i] = S - E[S_{\rho^*}] \geq (\frac{1}{2} - \epsilon)S$. Since $E[\sum_{i=\lceil\rho^* n\rceil+1}^{n} Y_i]$ is a summation of $n - \lceil\rho^* n\rceil$ terms, and $E[Y_{\lceil\rho^* n\rceil}] \geq E[Y_i]$ for all $i \geq \lceil\rho^* n\rceil$, then we have

$$E[Y_{\lceil\rho^* n\rceil}] \geq \frac{E[\sum_{i=\lceil\rho^* n\rceil+1}^{n} Y_i]}{n - \lceil\rho^* n\rceil} \geq \frac{(\frac{1}{2} - \epsilon)S}{n - \lceil\rho^* n\rceil} \geq \frac{(\frac{1}{2} - \epsilon)S}{n}. \tag{A.17}$$

Then for any $\rho < \rho^*$, for every $\epsilon > 0$, when $n$ is large enough,

$$\begin{aligned}
E[S_\rho] &= E[S_{\rho^*}] - \sum_{i=\lceil\rho n\rceil+1}^{\lceil\rho^* n\rceil} E[Y_i] \\
&\leq E[S_{\rho^*}] - (\lceil\rho^* n\rceil - \lceil\rho n\rceil)E[Y_{\lceil\rho^* n\rceil}] \\
&\leq (\frac{1}{2} + \epsilon)S - (\lceil\rho^* n\rceil - \lceil\rho n\rceil)\frac{(\frac{1}{2} - \epsilon)S}{n} \\
&\leq (\frac{1}{2} + \epsilon)S - (\rho^* - \rho - \frac{1}{n})(\frac{1}{2} - \epsilon)S
\end{aligned}$$

where the first inequality holds since each $Y_i$ with $i \leq \lceil\rho^* n\rceil$ has expectation at least as large as $E[Y_{\lceil\rho^* n\rceil}]$, and the second inequality follows from (A.16) and (A.17). Then for any $\rho < \rho^*$, we can pick $\epsilon > 0$ small enough such that $E[S_\rho]/S \leq (\frac{1}{2} + \epsilon) - (\rho^* - \rho - \frac{1}{n})(\frac{1}{2} - \epsilon) \leq \frac{1}{2} - 2\delta$ for a suitable $\delta > 0$ when $n$ is large enough. The result follows by combining the above with Lemma 2. □

## A.6 Proof of Lemma 3

*Proof.* For any given $\gamma > 0$, there exists a $\gamma$-net $\Sigma$ in $\mathcal{R}^{n-m}$ of cardinality less than $(1+\frac{2}{\gamma})^{n-m}$([88]). A $\gamma$-net $\Sigma$ is a set of points in $\mathcal{R}^{n-m}$ such that $\|\mathbf{v}^k\|_2 = 1$ for all $\mathbf{v}^k$ in $\Sigma$ and for any $\mathbf{z} \in \mathcal{R}^{n-m}$ with $\|\mathbf{z}\|_2 = 1$, there exists some $\mathbf{v}^k$ such that $\|\mathbf{z} - \mathbf{v}^k\|_2 \le \gamma$.

Since $B$ has i.i.d $\mathcal{N}(0,1)$ entries, then $B\mathbf{v}^k$ has $n$ i.i.d. $\mathcal{N}(0,1)$ entries for every $\mathbf{v}^k$. From Corollary 1 and 2, we know that given any $\rho < \rho^*$, for some $\delta > 0$ and for every $\epsilon > 0$, there exists $c_2 > 0$ and $c_3$ such that with probability at least $1 - 2e^{-c_2 n} - 2e^{-c_3 n}$, we have

$$S_\rho(B\mathbf{v}^k) \le (\frac{1}{2} - \delta)S \tag{A.18}$$

and

$$(1-\epsilon)S \le S_1(B\mathbf{v}^k) \le (1+\epsilon)S \tag{A.19}$$

both hold for *one* vector $\mathbf{v}^k$ in $\Sigma$. Then applying union bound, we know that (A.18) and (A.19) hold for *all* vectors in $\Sigma$ with probability at least

$$1 - (1 + 2/\gamma)^{n-m}(2e^{-c_2 n} + 2e^{-c_3 n}). \tag{A.20}$$

Let $\alpha = m/n$, then as long as $\alpha$ is large enough, say greater than $c_4 := 1 - \frac{\min(c_2, c_3)}{2\ln(1+2/\gamma)}$, then (A.20) is greater than $1 - e^{-c_5 n}$ for some constant $c_5 > 0$.

For any $\mathbf{z}$ such that $\|\mathbf{z}\|_2 = 1$, there exists $\mathbf{v}_0$ in $\Sigma$ such that $\|\mathbf{z} - \mathbf{v}_0\|_2 \triangleq \gamma_1 \le \gamma$. Let $\mathbf{z}_1$ denote $\mathbf{z} - \mathbf{v}_0$, then $\|\mathbf{z}_1 - \gamma_1\mathbf{v}_1\|_2 \triangleq \gamma_2 \le \gamma_1\gamma \le \gamma^2$ for some $\mathbf{v}_1$ in $\Sigma$. Repeating this process, we have

$$\mathbf{z} = \sum_{j\ge 0} \gamma_j \mathbf{v}_j \tag{A.21}$$

where $\gamma_0 = 1$, $\gamma_j \le \gamma^j$ and $\mathbf{v}_j \in \Sigma$. Thus for any $\mathbf{z} \in \mathcal{R}^{n-m}$, we have $\mathbf{z} = \|\mathbf{z}\|_2 \sum_{j\ge 0} \gamma_j \mathbf{v}_j$.

For any index set $T$ with $|T| \leq \lceil \rho n \rceil$,

$$\begin{aligned} \|B_T \mathbf{z}\|_p^p &= \|\mathbf{z}\|_2^p \|\sum_{j\geq 0} \gamma_j B_T \mathbf{v}_j\|_p^p \\ &\leq \|\mathbf{z}\|_2^p \sum_{j\geq 0} \gamma^{jp} \|B_T \mathbf{v}_j\|_p^p \\ &\leq S\|\mathbf{z}\|_2^p \frac{1-2\delta}{2(1-\gamma^p)}, \end{aligned}$$

where the first inequality holds from the triangular inequality and the fact that $\gamma_j \leq \gamma^j$. The second inequality holds with overwhelming probability by (A.18) and (A.20).

$$\begin{aligned} \|B\mathbf{z}\|_p^p &= \|\mathbf{z}\|_2^p \|\sum_{j\geq 0} \gamma_j B \mathbf{v}_j\|_p^p \\ &\geq \|\mathbf{z}\|_2^p (\|B v_0\|_p^p - \sum_{j\geq 1} \gamma_j^p \|B\mathbf{v}_j\|_p^p) \\ &\geq \|\mathbf{z}\|_2^p (\|B \mathbf{v}_0\|_p^p - \sum_{j\geq 1} \gamma^{jp} \|B\mathbf{v}_j\|_p^p) \\ &\geq \|\mathbf{z}\|_2^p ((1-\epsilon)S - \sum_{j\geq 1} \gamma^{jp}(1+\epsilon)S) \\ &\geq S\|\mathbf{z}\|_2^p \frac{1-2\gamma^p-\epsilon}{1-\gamma^p}, \end{aligned}$$

where the first inequality holds from the triangular inequality and the third inequality holds with overwhelming probability by (A.19) and (A.20). Thus $\|B_{T^c}\mathbf{z}\|_p^p - \|B_T\mathbf{z}\|_p^p \geq S\|\mathbf{z}\|_2^p \frac{2\delta-2\gamma^p-\epsilon}{1-\gamma^p}$ holds with probability at least $1 - e^{-c_5 n}$. For the given $\delta$ from Corollary 1, we can pick $\gamma$ and $\epsilon$ small enough such that $\|B_{T^c}\mathbf{z}\|_p^p - \|B_T\mathbf{z}\|_p^p \geq \delta S\|\mathbf{z}\|_2^p$. □

## A.7 Proof of Lemma 4

*Proof.* We first consider the case that $p = 0$. Now $\mu = E[|X|^0] = 1$, where $X \sim \mathcal{N}(0, 1)$. We have $\sum_{i\in T: X_i<0} |X_i|^p = \sum_{i\in T} \mathbf{1}_{\{X_i<0\}}$. Since $P(X_i < 0) = 0.5$ independently for all $i$ in $T$, then $E[\sum_{i\in T} \mathbf{1}_{\{X_i<0\}}] = \rho n/2$, and from the Chernoff bound, we have

$$P(\sum_{i\in T} \mathbf{1}_{\{X_i<0\}} \geq (1+\epsilon)\rho n/2) \leq e^{-\epsilon^2 \rho n/2},$$

and

$$P(\sum_{i\in T} \mathbf{1}_{\{X_i<0\}} \geq (1-\epsilon)\rho n/2) \leq e^{-\epsilon^2 \rho n/2}.$$

It is easy to see that with probability one $\sum_{i\in T^c} |X_i|^p = \sum_{i\in T^c} \mathbf{1}_{\{X_i\neq 0\}} = (1-\rho)n$ holds. Therefore Lemma 4 follows for $p = 0$.

Now we consider the case that $p \in (0,1)$. Let $\mathbf{X} = [X_1, ..., X_n]^T$. Let $S_{T^-}(\mathbf{X}) = \sum_{i\in T: X_i<0} |X_i|^p$. For any $\mathbf{X}$ and $\mathbf{X}'$,

$$\begin{aligned} & |S_{T^-}(\mathbf{X}) - S_{T^-}(\mathbf{X}')| \\ = \; & |\sum_{i\in T} |X_i|^p \mathbf{1}_{\{X_i<0\}} - \sum_{i\in T} |X_i'|^p \mathbf{1}_{\{X_i'<0\}}| \\ \leq \; & \sum_{i\in T} \left||X_i|^p \mathbf{1}_{\{X_i<0\}} - |X_i'|^p \mathbf{1}_{\{X_i'<0\}}\right| \\ \leq \; & \sum_{i\in T} |X_i - X_i'|^p, \end{aligned} \tag{A.22}$$

where the first inequality follows from the triangular inequality. To see why the second inequality holds, we consider three different cases. If both $X_i < 0$ and $X_i' < 0$ hold, then $\left||X_i|^p \mathbf{1}_{\{X_i<0\}} - |X_i'|^p \mathbf{1}_{\{X_i'<0\}}\right| = \left||X_i|^p - |X_i'|^p\right| \leq |X_i - X_i'|^p$ where the inequality holds since $p \in (0,1)$. If both $X_i$ and $X_i'$ are nonnegative, then clearly $\left||X_i|^p \mathbf{1}_{\{X_i<0\}} - |X_i'|^p \mathbf{1}_{\{X_i'<0\}}\right| = 0 \leq |X_i - X_i'|^p$. If only one of $X_i$ and $X_i'$ is negative, we assume $X_i < 0$ without loss of generality, then $\left||X_i|^p \mathbf{1}_{\{X_i<0\}} - |X_i'|^p \mathbf{1}_{\{X_i'<0\}}\right| = |X_i|^p \leq |X_i - X_i'|^p$, where the inequality holds since $X_i < 0$ and $X_i' \geq 0$. Combining the three cases, we know that $\left||X_i|^p \mathbf{1}_{\{X_i<0\}} - |X_i'|^p \mathbf{1}_{\{X_i'<0\}}\right| \leq |X_i - X_i'|^p$ always holds, thus the second inequality in (A.22) holds.

From the isoperimetric inequality for the Gaussian measure [88], for any set $A \in \mathcal{R}^n$ with measure at least a half, the set $A_t = \{\mathbf{x} \in \mathcal{R}^n : d(\mathbf{x}, A) \leq t\}$ has measure at least $1 - e^{-t^2/2}$, where $d(\mathbf{x}, A) = \inf_{\mathbf{y}\in A} \|\mathbf{x} - \mathbf{y}\|_2$. Let $M_{T^-}$ be the median value of $S_{T^-} = S_{T^-}(\mathbf{X})$. Define set $A = \{\mathbf{x} \in \mathcal{R}^n : S_{T^-}(\mathbf{x}) \leq M_{T^-}\}$, then

$$P(d(\mathbf{x}, A) \leq t) \geq 1 - e^{-t^2/2}.$$

We claim that $d(\mathbf{x}, A) \leq t$ implies that $S_{T^-}(\mathbf{x}) \leq M_{T^-} + (\rho n)^{(1-p/2)} t^p$. If $\mathbf{x} \in A$, then $S_{T^-}(\mathbf{x}) \leq M_{T^-}$, thus the claim holds as $(\rho n)^{1-p/2} t^p$ is nonnegative. If $\mathbf{x} \notin A$, then there exists $\mathbf{x}' \in A$ such that $\|\mathbf{x} - \mathbf{x}'\|_2 \leq t$. For $i$ in $T$, let $u_i = 1$ and let $v_i = |x_i - x_i'|^p$. From Hölder's inequality,

$$\begin{aligned}
\sum_{i\in T} |x_i - x_i'|^p &\leq \left(\sum_{i\in T} |u_i|^{2/(2-p)}\right)^{1-p/2} \left(\sum_{i\in T} |v_i|^{2/p}\right)^{p/2} \\
&\leq (\rho n)^{(1-p/2)} (t^2)^{p/2} = (\rho n)^{(1-p/2)} t^p \qquad \text{(A.23)}
\end{aligned}$$

From (A.22) and (A.23), $|S_{T^-}(\mathbf{x}) - S_{T^-}(\mathbf{x}')| \leq (\rho n)^{(1-p/2)} t^p$. Since $\mathbf{x} \notin A$ and $\mathbf{x}' \in A$, then $S_{T^-}(\mathbf{x}) > M_{T^-} \geq S_{T^-}(\mathbf{x}')$. Thus $S_{T^-}(\mathbf{x}) \leq M_{T^-} + (\rho n)^{(1-p/2)} t^p$, which verifies our claim. Then

$$\begin{aligned}
P(S_{T^-}(\mathbf{x}) &\leq M_{T^-} + (\rho n)^{(1-p/2)} t^p) \geq P(d(\mathbf{x}, A) \leq t) \\
&\geq 1 - e^{-t^2/2}. \qquad \text{(A.24)}
\end{aligned}$$

Similarly,

$$P(S_{T^-}(\mathbf{x}) \geq M_{T^-} - (\rho n)^{(1-p/2)} t^p) \geq 1 - e^{-t^2/2}. \qquad \text{(A.25)}$$

Combining (A.24) and (A.25),

$$P(|S_{T^-}(x) - M_{T^-}| \geq (\rho n)^{(1-p/2)} t^p) \leq 2e^{-t^2/2}. \qquad \text{(A.26)}$$

The difference of $E[S_{T^-}]$ and $M_{T^-}$ can be bounded as follows,

$$\begin{aligned}
|E[S_{T^-}] - M_{T^-}| &\leq E[|S_{T^-} - M_{T^-}|] \\
&= \int_0^\infty P(|S_{T^-}(\mathbf{x}) - M_{T^-}| \geq y) dy \\
&\leq \int_0^\infty 2e^{-\frac{1}{2} y^{\frac{2}{p}} (\rho n)^{(1-\frac{2}{p})}} dy \\
&= (\rho n)^{(1-\frac{p}{2})} \int_0^\infty 2e^{-\frac{1}{2} s^{\frac{2}{p}}} ds
\end{aligned}$$

Note that $c := \int_0^\infty 2e^{-\frac{1}{2}s^{(2/p)}}ds$ is a finite constant for all $p \in (0,1]$. Since $p > 0$, for any $\epsilon > 0$, $c(\rho n)^{(1-\frac{p}{2})} < \epsilon\rho n/4$ when $n$ is large enough.

Let $t = (\frac{\epsilon}{4})^{\frac{1}{p}}\sqrt{\rho n}$, from (A.26) with probability at least $1 - 2e^{-\frac{1}{2}(\frac{\epsilon}{4})^{\frac{2}{p}}\rho n}$, $|S_{T^-} - M_{T^-}| < \epsilon\rho n/4$. Thus $|S_{T^-} - E[S_{T^-}]| \leq |S_{T^-} - M_{T^-}| + |M_{T^-} - E[S_{T^-}]| < \epsilon\rho n/2$ holds with probability at least $1 - 2e^{-d_1 n}$ for some constant $d_1$. Since $E[S_{T^-}] = \mu\rho n/2$, where $\mu = E[|X|^p]$, then

$$\frac{1}{2}\rho n(\mu - \epsilon) < \sum_{i \in T: X_i < 0} |X_i|^p < \frac{1}{2}\rho n(\mu + \epsilon)$$

holds with probability at least $1 - 2e^{-d_1 n}$.

Similarly we can prove that with probability at least $1-2e^{-d_2 n}$ for some $d_2 > 0$,

$$(1-\rho)n(\mu - \epsilon) < \sum_{i \in T^c} |X_i|^p < (1-\rho)n(\mu + \epsilon)$$

holds. Then by a simple union bound, the above two statements hold at the same time with probability at least $1 - 2e^{-d_2 n} - 2e^{-d_2 n}$, thus Lemma 4 follows. □

## A.8 Proof of Theorem 6

*Proof.* From Lemma 4, applying similar arguments in the proof of Lemma 3, we get that when $\alpha > c_7$ for some $0 < c_7 < 1$ and $n$ is large enough, with probability $1 - e^{-c_8 n}$ for some $c_8 > 0$,

- $\frac{1}{2}\rho n(\mu - \epsilon) < \sum_{i \in T: B_i\mathbf{v} < 0} |B_i\mathbf{v}|^p < \frac{1}{2}\rho n(\mu + \epsilon)$
- $(1-\rho)n(\mu - \epsilon) < \sum_{i \in T^c} |B_i\mathbf{v}|^p < (1-\rho)n(\mu + \epsilon)$

hold for all the vectors $\mathbf{v}$ in a $\gamma$-net $\Sigma$ at the same time. Let $\mathcal{S}$ be the unit sphere in $\mathcal{R}^{n-m}$. Pick any $\mathbf{z} \in \mathcal{S}$, from (A.21) we have $\mathbf{z} = \sum_{j \geq 0} \gamma_j \mathbf{v}_j$, where $\gamma_0 = 1$, $\mathbf{v}_j \in \Sigma$

for all $j$ and $\gamma_j \leq \gamma^j$.

Given $\mathbf{z}$, let $T^- = \{i \in T : B_i\mathbf{z} < 0\}$. For any $i$ in $T^-$,

$$\begin{aligned}
|B_i\mathbf{z}|^p &= \Big|\sum_{j\geq 0}\gamma_j B_i\mathbf{v}_j\Big|^p \\
&= \Big|\sum_{j:B_i\mathbf{v}_j<0}\gamma_j B_i\mathbf{v}_j + \sum_{j:B_i\mathbf{v}_j\geq 0}\gamma_j B_i\mathbf{v}_j\Big|^p \\
&\leq \Big|\sum_{j:B_i\mathbf{v}_j<0}\gamma_j B_i\mathbf{v}_j\Big|^p \\
&\leq \sum_{j:B_i\mathbf{v}_j<0}\gamma^{jp}|B_i\mathbf{v}_j|^p
\end{aligned}$$

where the first inequality holds as $B_i\mathbf{z} < 0$. Then

$$\begin{aligned}
\|B_{T^-}\mathbf{z}\|_p^p &\leq \sum_{i\in T^-}\sum_{j:B_i\mathbf{v}_j<0}\gamma^{jp}|B_i\mathbf{v}_j|^p \\
&\leq \sum_{i\in T}\sum_{j:B_i\mathbf{v}_j<0}\gamma^{jp}|B_i\mathbf{v}_j|^p \\
&= \sum_{j\geq 0}\gamma^{jp}\sum_{i\in T:B_i\mathbf{v}_j<0}|B_i\mathbf{v}_j|^p \qquad (A.27) \\
&< \frac{1}{2(1-\gamma^p)}\rho n(\mu+\epsilon), \qquad (A.28)
\end{aligned}$$

where the last inequality holds with overwhelming probability.

We also have

$$\begin{aligned}
&\|B_{T^c}\mathbf{z}\|_p^p = \|(\sum_{j\geq 0}\gamma_j B_{T^c}\mathbf{v}_j)\|_p^p \\
\geq\ & \|B_{T^c}\mathbf{v}_0\|_p^p - \sum_{j\geq 1}\gamma^{jp}\|B_{T^c}\mathbf{v}_j\|_p^p \\
>\ & (1-\rho)n(\mu-\epsilon) - \sum_{j\geq 1}\gamma^{jp}(1-\rho)n(\mu+\epsilon) \\
\geq\ & (1-\rho)n\frac{\mu-2\mu\gamma^p-\epsilon}{1-\gamma^p}, \qquad (A.29)
\end{aligned}$$

where the second inequality holds with overwhelming probability.

Combining (A.28) and (A.29), we have for every $\mathbf{z} \in \mathcal{S}$, $\|B_{T^c}\mathbf{z}\|_p^p - \|B_{T^-}\mathbf{z}\|_p^p > \frac{n\mu}{1-\gamma^p}(1 - \frac{3}{2}\rho - 2\gamma^p(1-\rho) - \frac{\epsilon}{\mu}(1-\frac{\rho}{2}))$ holds at the same time with overwhelming probability. Then with overwhelming probability, for every non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$, we have $\|B_{T^c}\mathbf{z}\|_p^p - \|B_{T^-}\mathbf{z}\|_p^p > \|\mathbf{z}\|_2^p \frac{n\mu}{1-\gamma^p}(1 - \frac{3}{2}\rho - 2\gamma^p(1-\rho) - \frac{\epsilon}{\mu}(1-\frac{\rho}{2}))$. For any $\rho < \frac{2}{3}$, we can pick $\gamma$ and $\epsilon$ small enough such that the righthand side is positive. The result follows by applying Theorem 3 and Theorem 4. □

## A.9 Upper bound of $\|B\mathbf{z}\|_p^p$ for all $\mathbf{z} \in \mathcal{S}$

**Lemma 13.** *Given any $\alpha$ and $p$, there exists a constant $\lambda_{\max}(\alpha, p) > 0$ and some constant $c_{16} > 0$ such that with probability at least $1 - e^{-c_{16}n}$, for every $\mathbf{z} \in \mathcal{S}$, $\|B\mathbf{z}\|_p^p < \lambda_{\max}(\alpha, p)n$.*

To help improve the lower bound of the recovery threshold, we would like $\lambda_{\max}(\alpha, p)$ to be as small as possible, while at the same time, the probability that $\|B\mathbf{z}\|_p^p$ exceeds $\lambda_{\max}(\alpha, p)n$ for some $\mathbf{z}$ in $\mathcal{S}$ still has exponential decay to zero. Therefore, in the following proof, besides establishing the existence of $\lambda_{\max}(\alpha, p)$, we make some efforts to reduce the value of $\lambda_{\max}(\alpha, p)$, and $\lambda_{\max}(\alpha, p)$ can be computed following the lines and finally through (A.34).

*Proof.* Define $c_{\max} = \frac{1}{n}\max_{\mathbf{z}\in\mathcal{S}} \|B\mathbf{z}\|_p^p$, then for any non-zero vector $\mathbf{z}$, $\|B\mathbf{z}\|_p^p \leq \|\mathbf{z}\|_p^p c_{\max} n$. Let $\Sigma_1$ be a $\gamma$-net of $\mathcal{S}$ with cardinality at most $(1 + 2/\gamma)^{n-m}$ [88] and $\gamma > 0$ to be chosen later, and define

$$\eta = \frac{1}{n}\max_{\mathbf{z}\in\Sigma_1} \|B\mathbf{z}\|_p^p.$$

Then from the definition of $\gamma$-net, for every $\mathbf{z} \in \mathcal{S}$, there exists $\mathbf{z}' \in \Sigma_1$ such that $\|\mathbf{z} - \mathbf{z}'\|_2 \leq \gamma$. Note that for every $\mathbf{z} \in \mathcal{S}$, $\|B\mathbf{z}\|_p^p \leq \|B\mathbf{z}'\|_p^p + \|B(\mathbf{z} - \mathbf{z}')\|_p^p = \|B\mathbf{z}'\|_p^p + \|\mathbf{z} - \mathbf{z}'\|_2^p \|B\frac{\mathbf{z}-\mathbf{z}'}{\|\mathbf{z}-\mathbf{z}'\|_2}\|_p^p \leq \eta n + \gamma^p c_{\max} n$, where the first inequality follows from the triangular inequality and the second inequality follows from the definition of $\eta$ and $c_{\max}$. Then $c_{\max}n \leq \eta n + \gamma^p c_{\max}n$, which leads to

$$c_{\max} \leq \eta/(1-\gamma^p). \tag{A.30}$$

To characterize $c_{\max}$, we first characterize $\eta$. For any $a > E[|X|^p]$ where $X \sim \mathcal{N}(0,1)$, we calculate the probability that $\|B\mathbf{z}\|_p^p \geq an$ for some $\mathbf{z}$ in $\Sigma_1$. Note that $\forall \mathbf{z} \in \mathcal{S}$, $B_i\mathbf{z}$ $(i = 1, ..., n)$ are i.i.d. $\mathcal{N}(0,1)$ random variables where $B_i$ is the $i^{\text{th}}$ row of $B$. Then

$$\begin{aligned} & P(\eta \geq a) = P(\exists \mathbf{z} \in \Sigma_1 \text{s.t. } \|B\mathbf{z}\|_p^p \geq an) \\ \leq\ & \sum_{\mathbf{z}\in\Sigma_1} P(\|B\mathbf{z}\|_p^p \geq an) \\ \leq\ & (1+2/\gamma)^{n-m} \min_{t>0} e^{-tan} E[e^{t\sum_i |B_i\mathbf{z}|^p}] \\ =\ & (1+2/\gamma)^{(1-\alpha)n} \min_{t>0} e^{-tan} (E[e^{t|X|^p}])^n \\ =\ & e^{((1-\alpha)\log(1+\frac{2}{\gamma})+\min_{t>0}(\log(E[e^{t|X|^p}])-at))n}, \end{aligned} \tag{A.31}$$

where $X \sim \mathcal{N}(0,1)$, the first inequality follows from the union bound, and the second inequality follows from the Chernoff bound.

To obtain a good upper bound of $\eta$, we would like to find the smallest $a$ such that the upper bound of $P(\eta \geq a)$ in (A.31) still exponentially decays to zero, note that we do not care about the decay rate here. To solve the minimization problem in the righthand side of (A.31), note that $\log(E[e^{t|X|^p}])$ is the cumulant generating function and is known to be convex [45] with respect to $t$, then $\log(E[e^{t|X|^p}]) - at$ is also convex, and its minimum is achieved where its first derivative with respect to $t$ is 0. Define $t^* := \arg\min_t[\log(E[e^{t|X|^p}]) - at]$, then we have

$$
\begin{aligned}
0 &= \frac{d[\log(E[e^{t|X|^p}]) - at]}{dt}\Big|_{t=t^*} \\
&= \frac{E[|X|^p e^{t^*|X|^p}]}{E[e^{t^*|X|^p}]} - a. \qquad \text{(A.32)}
\end{aligned}
$$

(A.32) determines $t^*$ given $a$. The derivative of $t^*$ with respect to $a$ is

$$
\begin{aligned}
\frac{dt^*}{da} &= \left(\frac{da}{dt^*}\right)^{-1} \\
&= \frac{(E[e^{t^*|X|^p}])^2}{E[e^{t^*|X|^p}]E[|X|^{2p}e^{t^*|X|^p}] - (E[|X|^p e^{t^*|X|^p}])^2},
\end{aligned}
$$

Note that $(E[|X|^p e^{t^*|X|^p}])^2 = (E[e^{t^*|X|^p/2} \cdot (|X|^p e^{t^*|X|^p/2})])^2 < E[e^{t^*|X|^p}]E[|X|^{2p}e^{t^*|X|^p}]$, where the inequality follows from Cauchy-Schwarz inequality and the fact that the functions $e^{t^*|X|^p}$ and $|X|^{2p}e^{t^*|X|^p}$ are not linearly dependent. Thus, $\frac{dt^*}{da} > 0$. Since when $a = E[|X|^p]$, we have $t^* = 0$ from (A.32), then when $a > E[|X|^p]$ we have $t^* > 0$. Thus when $a > E[|X|^p]$, it holds that $t^* = \arg\min_{t>0}(\log(E[e^{t|X|^p}]) - at)$. Given $a$, we can numerically compute $t^*$ by (A.32) and plug it into (A.31) to obtain an upper bound of $P(\eta \geq a)$. Then the question is how small can $a$ be while the exponent on the righthand side of (A.31) is still negative. Note that given $\gamma$, the exponent on the righthand side of (A.31) is negative when $a$ is large enough. To see this, if we let $t = 2(1-\alpha)\log(1+2/\gamma)/a$, then $\log(E[e^{t|X|^p}]) - at$ goes to $-2(1-\alpha)\log(1+2/\gamma)$ as $a$ goes to infinity. Thus, when $a$ is sufficiently large, $\min_{t>0}\log(E[e^{t|X|^p}]) - at < -(1-\alpha)\log(1+2/\gamma) < 0$, in other words, the exponent on the righthand side of (A.31) is negative. Pick $\hat{a}(\alpha, p, \gamma)$ such that the exponent on the righthand side of (A.31) is negative for all $a \geq \hat{a}(\alpha, p, \gamma)$, and positive for all $a \leq \hat{a}(\alpha, p, \gamma) - \epsilon$ for a very small $\epsilon > 0$. Therefore

$$
(1-\alpha)\log(1+\frac{2}{\gamma}) + \min_{t>0}(\log(E[e^{t|X|^p}]) - \hat{a}(\alpha, p, \gamma)t) < 0. \qquad \text{(A.33)}
$$

Then there exists some constant $c_{16} > 0$ such that

$$\begin{aligned} & P(\eta \geq \hat{a}(\alpha, p, \gamma)) \\ \leq \ & e^{((1-\alpha)\log(1+\frac{2}{\gamma})+\min_{t>0}(\log(E[e^{t|X|^p}])-\hat{a}(\alpha,p,\gamma)t))n} \\ = \ & e^{-c_{16}n}. \end{aligned}$$

Then the probability that $||B\mathbf{z}||_p^p \geq \frac{\hat{a}(\alpha,p,\gamma)n}{1-\gamma^p}$ holds for some $\mathbf{z} \in \mathcal{S}$ is

$$\begin{aligned} P(\max_{\mathbf{z}\in\mathcal{S}} ||B\mathbf{z}||_p^p \geq \frac{\hat{a}(\alpha, p, \gamma)n}{1-\gamma^p}) &= P(c_{\max} \geq \frac{\hat{a}(\alpha, p, \gamma)}{1-\gamma^p}) \\ &\leq P(\frac{\eta}{1-\gamma^p} \geq \frac{\hat{a}(\alpha, p, \gamma)}{1-\gamma^p}) \\ &\leq e^{-c_{16}n}, \end{aligned}$$

where the first inequality follows from (A.30). Thus, for all $\gamma \in (0, 1)$, $\hat{a}(\alpha, p, \gamma)n/(1-\gamma^p)$ can be viewed as an upper bound of $||B\mathbf{z}||$ for all $\mathbf{z} \in \mathcal{S}$ in the sense that the probability that $||B\mathbf{z}||_p^p \geq \hat{a}(\alpha, p, \gamma)n/(1-\gamma^p)$ for some $\mathbf{z} \in \mathcal{S}$ decays exponentially to zero for every $\gamma$ in $(0, 1)$. Since we would like such an upper bound to be as small as possible, we let

$$\lambda_{\max}(\alpha, p) = \min_{\gamma\in(0,1)} \hat{a}(\alpha, p, \gamma)/(1-\gamma^p), \tag{A.34}$$

then with probability at least $1-e^{-c_{16}n}$ for some $c_{16}(\alpha, p, \lambda_{\max}) > 0$, for every $\mathbf{z} \in \mathcal{S}$, $||B\mathbf{z}||_p^p < \lambda_{\max} n$ holds. Thus, the statement follows. □

## A.10 Calculation of $\lambda_{\min}(\alpha, p)$ in Lemma 5

Given $\alpha$ and $p$, define

$$c_{\max} = \frac{1}{n}\sup_{\mathbf{z}\in\mathcal{S}} ||B\mathbf{z}||_p^p = \frac{1}{n}\max_{\mathbf{z}\in\mathcal{S}} ||B\mathbf{z}||_p^p,$$

where the second equality holds by compactness. Thus, for any vector $\mathbf{z}$, $||B\mathbf{z}||_p^p \leq ||\mathbf{z}||_p^p c_{\max} n$. Define

$$c_{\min} = \frac{1}{n} \min_{\mathbf{z} \in \mathcal{S}} ||B\mathbf{z}||_p^p.$$

Pick a $\gamma$-net $\Sigma_2$ of $\mathcal{S}$ with cardinality at most $(1 + 2/\gamma)^{n-m}$ [88] and $\gamma > 0$ to be chosen later, we define

$$\theta = \frac{1}{n} \min_{\mathbf{z} \in \Sigma_2} ||B\mathbf{z}||_p^p.$$

Then for every $\mathbf{z} \in \mathcal{S}$, there exists $\mathbf{z}' \in \Sigma_2$ such that $||\mathbf{z} - \mathbf{z}'||_2 \leq \gamma$. We have

$$||B\mathbf{z}||_p^p \geq ||B\mathbf{z}'||_p^p - ||B(\mathbf{z} - \mathbf{z}')||_p^p \geq \theta n - \gamma^p c_{\max} n, \tag{A.35}$$

where the first inequality follows from triangular inequality and the second inequality follows from the definition of $c_{\max}$. Since (A.35) holds for every $\mathbf{z}$ in $\mathcal{S}$, we have

$$c_{\min} \geq \theta - \gamma^p c_{\max}. \tag{A.36}$$

We aim to find a value $\lambda_{\min}(\alpha, p)$ as large as possible such that $c_{\min} > \lambda_{\min}(\alpha, p)$ still holds with overwhelming probability. We will calculate a "lower bound" of $\theta$ and an "upper bound" of $c_{\max}$, and then obtain a "lower bound" of $c_{\min}$ by (A.36).

We first consider the lower bound of $\theta$. For any constant $b > 0$, we will calculate the probability that $\theta$ is less than $b$. We want to obtain a value $b$ large enough but this probability still decays exponentially to 0. And we treat such a value as the lower bound of $\theta$. Given any constant $b > 0$,

$$
\begin{aligned}
& P(\theta \le b) = P(\exists \mathbf{z} \in \Sigma_2 \text{ s.t. } \|B\mathbf{z}\|_p^p \le bn) \\
\le \quad & \sum_{\mathbf{z} \in \Sigma_2} P(\|B\mathbf{z}\|_p^p \le bn) \\
\le \quad & (1 + 2/\gamma)^{n-m} e^{tbn} E[e^{-t \sum_i |B_i \mathbf{z}|^p}], \quad \forall t > 0 \\
= \quad & (1 + 2/\gamma)^{(1-\alpha)n} e^{tbn} E[e^{-t|X|^p}]^n, \quad \forall t > 0 \\
= \quad & e^{((1-\alpha)\log(1+2/\gamma) + \log(E[e^{-t|X|^p}]) + bt)n}, \quad \forall t > 0, \qquad \text{(A.37)}
\end{aligned}
$$

where $X \sim \mathcal{N}(0, 1)$, and the first inequality follows from the union bound. The second inequality follows from the Chernoff bound and the fact that $P(\|B\mathbf{z}\|_p^p \le bn)$ is the same for all $\mathbf{z} \in \Sigma_2$ since $B$ has i.i.d. $\mathcal{N}(0, 1)$ entries. Note that

$$
\begin{aligned}
E[e^{-t|X|^p}] &= \sqrt{2/\pi} \int_0^\infty e^{-tx^p} e^{-\frac{1}{2}x^2} dx \\
&= t^{-\frac{1}{p}} \sqrt{2/\pi} \int_0^\infty e^{-y^p} e^{-\frac{1}{2}(t^{-\frac{1}{p}} y)^2} dy. \qquad \text{(A.38)} \\
&\le t^{-\frac{1}{p}} \sqrt{2/\pi} \int_0^\infty e^{-y^p} dy \\
&= t^{-\frac{1}{p}} \sqrt{2/\pi}\, \Gamma(1/p)/p, \qquad \text{(A.39)}
\end{aligned}
$$

where (A.38) holds from changing variables using $x = t^{-\frac{1}{p}} y$, and the inequality follows from the fact that $e^{-\frac{1}{2}(t^{-\frac{1}{p}} y)^2} \le 1$ for all $y \ge 0$. When $t > 1$, then $t^{-\frac{1}{p}} < 1$, then from (A.38) we have

$$
E[e^{-t|X|^p}] \ge t^{-\frac{1}{p}} \sqrt{2/\pi} \int_0^\infty e^{-y^p - \frac{1}{2}y^2} dy.
$$

Since $\int_0^\infty e^{-y^p - \frac{1}{2}y^2} dy$ exists and is positive, then combining (A.39) and (A.40), we have when $t > 1$,

$$
E[e^{-t|X|^p}] = \Theta(t^{-\frac{1}{p}}). \qquad \text{(A.40)}
$$

Since (A.37) holds for all $t > 0$, we let $t = \gamma^{-p(1-\alpha+\epsilon)} > 1$ for any $\epsilon$ such that $0 < \epsilon \le \alpha$ and let $b = 1/t$, then from (A.37) we have

$$
P(\theta \le \gamma^{p(1-\alpha+\epsilon)}) \le e^{((1-\alpha)\log(1+2/\gamma) + \log(\Theta(\gamma^{1-\alpha+\epsilon})) + 1)n}.
$$

Note that since $\epsilon > 0$, when $\gamma$ is sufficiently small, we have

$$(1-\alpha)\log(1+\frac{2}{\gamma}) + \log(\Theta(\gamma^{1-\alpha+\epsilon})) + 1 < 0. \tag{A.41}$$

Therefore when $\gamma \leq \xi$ for some small enough $\xi > 0$, there exists $\kappa > 0$ (depending on $\gamma$ and $\epsilon$) such that

$$P(\theta \leq \gamma^{p(1-\alpha+\epsilon)}) \leq e^{-\kappa n}. \tag{A.42}$$

Thus, for every $\epsilon \in (0, \alpha)$ and for all $\gamma \leq \xi$ with some $\xi$ depending on $\epsilon$, the probability that $\theta \leq \gamma^{p(1-\alpha+\epsilon)}$ decays exponentially to zero, though the decaying rate depends on $\epsilon$ and $\gamma$.

Lemma 13 indicates that there exists $\lambda_{\max}(\alpha, p)$ and $c_{16} > 0$ such that

$$P(c_{\max} < \lambda_{\max}(\alpha, p)) \geq 1 - e^{-c_{16}n}. \tag{A.43}$$

Then after characterizing $\theta$ and $c_{\max}$ separately, we are ready to characterize $c_{\min}$. We have

$$\begin{aligned}
& P(c_{\min} \leq \gamma^{p(1-\alpha+\epsilon)} - \gamma^p \lambda_{\max}(\alpha, p)) \\
\leq \; & P(\theta - \gamma^p c_{\max} \leq \gamma^{p(1-\alpha+\epsilon)} - \gamma^p \lambda_{\max}(\alpha, p)) \\
\leq \; & P(\theta \leq \gamma^{p(1-\alpha+\epsilon)}) + P(c_{\max} \geq \lambda_{\max}(\alpha, p)) \\
\leq \; & e^{-\kappa n} + e^{-c_{12}n},
\end{aligned}$$

where the first inequality follows from (A.36), and the last inequality follows from (A.42) and (A.43). Then for every $\epsilon \in (0, \alpha)$, for all $\gamma \leq \xi(\epsilon)$, there exists constant $c_9 > 0$ (depending on $\epsilon$ and $\gamma$) such that $P(c_{\min} \leq \gamma^{p(1-\alpha+\epsilon)} - \gamma^p \lambda_{\max}(\alpha, p)) \leq e^{-c_9 n}$. Given $\lambda_{\max}(\alpha, p)$, let

$$\lambda_{\min}(\alpha, p) = \max_{0<\epsilon<\alpha, 0<\gamma\leq\xi(\epsilon)} \gamma^{p(1-\alpha+\epsilon)} - \gamma^p \lambda_{\max}(\alpha, p). \tag{A.44}$$

Note that since $1 - \alpha + \epsilon < 1$, $\gamma^{p(1-\alpha+\epsilon)} - \gamma^p \lambda_{\max} > 0$ when $\gamma$ is sufficiently small, therefore $\lambda_{\min} > 0$, and Lemma 5 follows.

## A.11 Calculation of $\rho^*(\alpha, p)$ in Lemma 6

For any given set $T \subset \{1, 2, ..., n\}$ with $|T| = \rho n$ $(0 < \rho < 1)$, define

$$d_{\max} = \frac{1}{n} \max_{\mathbf{z} \in \mathcal{S}} \|B_T \mathbf{z}\|_p^p.$$

Since $B$ has i.i.d. Gaussian entries, then the distribution of $d_{\max}$ is the same for any $T$ with $|T| = \rho n$. Given a $\gamma$-net $\Sigma_3$ of $\mathcal{S}$ with cardinality at most $(1 + 2/\gamma)^{n-m}$ and $\gamma > 0$ to be chosen later, define

$$\tau = \frac{1}{n} \max_{\mathbf{z} \in \Sigma_3} \|B_T \mathbf{z}\|_p^p.$$

Then for every $\mathbf{z} \in \mathcal{S}$, there exists $\mathbf{z}' \in \Sigma_3$ such that $\|\mathbf{z} - \mathbf{z}'\|_2 \leq \gamma$. Then for every $\mathbf{z} \in \mathcal{S}$, we have $\|B_T \mathbf{z}\|_p^p \leq \|B_T \mathbf{z}'\|_p^p + \|B_T(\mathbf{z} - \mathbf{z}')\|_p^p \leq \tau n + \gamma^p d_{\max} n$. That means $d_{\max} n \leq \tau n + \gamma^p d_{\max} n$, which implies

$$d_{\max} \leq \tau/(1 - \gamma^p). \tag{A.45}$$

Given $\lambda_{\min}(\alpha, p)$ (denoted by $\lambda_{\min}$ here for simplicity), in order to obtain $\rho^*(\alpha, p)$ in Lemma 6, we essentially need to find the largest $\rho$ such that the probability that $d_{\max} \geq \lambda_{\min}/2$ holds for some support $T$ with $|T| = \rho n$ can still decay exponentially to 0. From (A.45), we first consider the probability that

$\tau \geq \lambda_{\min}(1-\gamma^p)/2$ holds for a given set $T$.

$$\begin{aligned}
& P(\tau \geq \lambda_{\min}(1-\gamma^p)/2,\ \text{given } T) \\
= \ & P(\exists \mathbf{z} \in \Sigma_3 \text{ s.t. } \|B_T\mathbf{z}\|_p^p \geq \lambda_{\min}(1-\gamma^p)n/2) \\
\leq \ & \sum_{\mathbf{z}\in\Sigma_3} P(\|B_T\mathbf{z}\|_p^p \geq \frac{\lambda_{\min}(1-\gamma^p)n}{2}) \\
= \ & \sum_{\mathbf{z}\in\Sigma_3} P(\sum_{i\in T} |B_i\mathbf{z}|^p \geq \frac{\lambda_{\min}(1-\gamma^p)n}{2}) \\
\leq \ & (1+2/\gamma)^{n-m} \min_{t>0} e^{-t\lambda_{\min}(1-\gamma^p)n/2} E[e^{t\sum_{i\in T}|B_i\mathbf{z}|^p}] \\
= \ & (1+2/\gamma)^{(1-\alpha)n} \min_{t>0} e^{-t\lambda_{\min}(1-\gamma^p)n/2} (E[e^{t|X|^p}])^{\rho n} \\
= \ & e^{((1-\alpha)\log(1+\frac{2}{\gamma})+\min_{t>0}(\rho\log(E[e^{t|X|^p}])-t\lambda_{\min}(1-\gamma^p)/2))n}, \qquad \text{(A.46)}
\end{aligned}$$

where $X \sim \mathcal{N}(0,1)$, the first inequality follows from the union bound and the second inequality follows from the Chernoff bound. Note that since $B$ has i.i.d. $\mathcal{N}(0,1)$ entries, (A.46) holds for any $T$ as long as $|T| = \rho n$.

Now consider the probability that $\|B_T\mathbf{z}\|_p^p \geq \frac{1}{2}\lambda_{\min}n$ for some $\mathbf{z} \in \mathcal{S}$ and $T$ with $|T| = \rho n$.

$$\begin{aligned}
& P(\exists \mathbf{z} \in \mathcal{S}, \exists T,\ \text{s.t. } |T| = \rho n, \|B_T\mathbf{z}\|_p^p \geq \lambda_{\min}n/2) \\
\leq \ & \binom{n}{\rho n} P(\exists \mathbf{z} \in \mathcal{S} \text{ s.t. } \|B_T\mathbf{z}\|_p^p \geq \lambda_{\min}n/2, \\
& \quad \text{for given } T \subset \{1,2,...,n\} \text{ and } |T| = \rho n) \\
= \ & \binom{n}{\rho n} P(d_{\max} \geq \lambda_{\min}/2,\ \text{given } T) \\
\leq \ & \binom{n}{\rho n} P(\tau/(1-\gamma^p) \geq \lambda_{\min}/2,\ \text{given } T) \\
= \ & \binom{n}{\rho n} P(\tau \geq \lambda_{\min}(1-\gamma^p)/2,\ \text{given } T) \\
\leq \ & 2^{nH(\rho)} e^{((1-\alpha)\log(1+2/\gamma)+\min_{t>0}(\rho\log(E[e^{t|X|^p}])-t\lambda_{\min}(1-\gamma^p)/2))n} \\
= \ & e^{(H(\rho)\log 2+(1-\alpha)\log(1+2/\gamma)+\min_{t>0}(\rho\log(E[e^{t|X|^p}])-t\lambda_{\min}(1-\gamma^p)/2))n}, \qquad \text{(A.47)}
\end{aligned}$$

where the first inequality follows from the union bound, the second inequality follows from (A.45), and the third inequality follows from (A.46) and the fact that $\binom{n}{\rho n} \leq 2^{nH(\rho)}$, where $H(\rho) = -\rho\log(\rho) - (1-\rho)\log(1-\rho)$.

To obtain a good upper bound of $P(\exists \mathbf{z} \in \mathcal{S}, \exists T, \text{s.t. } |T| \leq \rho n, \|B_T\mathbf{z}\|_p^p \geq \lambda_{\min} n/2)$, we first would like to solve the minimization problem on the righthand side of (A.47). Note that $\log(E[e^{t|X|^p}])$ is the cumulant generating function and is known to be convex [45] with respect to $t$. Then $\rho\log(E[e^{t|X|^p}]) - t\lambda_{\min}(1-\gamma^p)/2$ is also convex, then its minimum is achieved where its first derivative with respect to $t$ is 0. Define $t^* := \arg\min_t[\rho\log(E[e^{t|X|^p}]) - t\lambda_{\min}(1-\gamma^p)/2]$. We have

$$\begin{aligned} 0 &= \frac{d[\rho\log(E[e^{t|X|^p}]) - t\lambda_{\min}(1-\gamma^p)/2]}{dt}\Big|_{t=t^*} \\ &= \frac{\rho E[|X|^p e^{t^*|X|^p}]}{E[e^{t^*|X|^p}]} - \lambda_{\min}(1-\gamma^p)/2, \end{aligned}$$

which is equivalent to

$$\rho = \frac{\lambda_{\min}(1-\gamma^p)E[e^{t^*|X|^p}]}{2E[|X|^p e^{t^*|X|^p}]}. \tag{A.48}$$

(A.48) determines $t^*$ given $\rho$, $\lambda_{\min}$ and $\gamma$. The derivative of $t^*$ with respect to $\rho$ is

$$\begin{aligned} &\frac{dt^*}{d\rho} = (\frac{d\rho}{dt^*})^{-1} = \\ &\frac{2(E[|X|^p e^{t^*|X|^p}])^2}{\lambda_{\min}(1-\gamma^p)((E[|X|^p e^{t^*|X|^p}])^2 - E[e^{t^*|X|^p}]E[|X|^{2p} e^{t^*|X|^p}])}. \end{aligned}$$

Note that $(E[|X|^p e^{t^*|X|^p}])^2 = (E[e^{t^*|X|^p/2} \cdot (|X|^p e^{t^*|X|^p/2})])^2 < E[e^{t^*|X|^p}]E[|X|^{2p} e^{t^*|X|^p}]$, where the inequality follows from Cauchy-Schwarz inequality and the fact that functions $e^{t^*|X|^p}$ and $|X|^{2p}e^{t^*|X|^p}$ are not linearly dependent. Therefore from (A.49) we know $\frac{dt^*}{d\rho} < 0$. Since when $\rho = \lambda_{\min}(1-\gamma^p)/(2E[|X|^p])$, one can obtain from (A.48) that $t^* = 0$, therefore when $\rho < \lambda_{\min}(1-\gamma^p)/(2E[|X|^p])$, the corresponding $t^*$ is always positive. Thus, when $\rho < \lambda_{\min}(1-\gamma^p)/(2E[|X|^p])$, $t^*$ defined in (A.48) is the solution to $\min_{t>0}(\rho\log(E[e^{t|X|^p}]) - t\lambda_{\min}(1-\gamma^p)/2)$. Given $\rho$, $\gamma$, and $\alpha$, we can numerically compute $t^*$ by (A.48) and plug it into (A.47) to obtain an upper bound of $P(\exists \mathbf{z} \in \mathcal{S}, \exists T, \text{ s.t. } |T| \leq \rho n, \|B_T\mathbf{z}\|_p^p \geq \lambda_{\min} n/2)$.

Now that given $\alpha$ and $\lambda_{\min}$, for any $\rho$, (A.47) provides an upper bound of the probability that there exists some $\mathbf{z} \in \mathcal{S}$ and some $T$ with $|T| = \rho n$ such that $\|B_T \mathbf{z}\|_p^p \geq \lambda_{\min} n/2$ holds. The next question is how large $\rho$ could be such that this upper bound still decays exponentially to zero. The largest $\rho$ is indeed the $\rho^*(\alpha, p)$ we would like to calculate.

Note that given $\alpha$, $p$, and $\lambda_{\min}$, for every $\gamma$, as $\rho$ goes to 0, $H(\rho)$ goes to 0, and $\min_{t>0}(\rho \log(E[e^{t|X|^p}]) - t\lambda_{\min}(1-\gamma^p)/2$ goes to $-\infty$, thus, there exists $\hat{\rho}(\alpha, p, \gamma) > 0$ such that the exponent on the righthand side of (A.47) is negative for all $\rho \leq \hat{\rho}(\alpha, p, \gamma)$, and is positive for all $\rho > \hat{\rho}(\alpha, p, \gamma) + \epsilon$ for some very small $\epsilon > 0$. In other words, for each $\gamma$, $P(\exists \mathbf{z} \in \mathcal{S}, \exists T, \text{ s.t. } |T| = \hat{\rho}(\alpha, p, \gamma)n, \|B_T \mathbf{z}\|_p^p \geq \lambda_{\min} n/2) \leq e^{-cn}$ for some positive $c$ depending on $\gamma$. We then optimize $\hat{\rho}(\alpha, p, \gamma)$ over $\gamma \in (0, 1)$, and let

$$\rho^*(\alpha, p) = \max_{\gamma \in (0,1)} \hat{\rho}(\alpha, p, \gamma),$$

then with probability at least $1 - e^{-c_{10} n}$ for some $c_{10} > 0$, for every $\mathbf{z} \in \mathcal{S}$ and for every set $T \subset \{1, 2, ..., n\}$ with $|T| \leq \rho^*(\alpha, p)n$, $\|B_T \mathbf{z}\|_p^p < \lambda_{\min} n/2$ holds simultaneously. Then Lemma 6 follows.

## A.12 Proof of Theorem 7

*Proof.* Let $\mathcal{S}$ be the unit sphere in $\mathcal{R}^{n-m}$. Then

$$\begin{aligned}
& P(\text{Strong recovery succeeds to recover} \\
& \text{ vectors up to } \rho^*(\alpha, p)n\text{-sparse}) \\
= \; & P(\forall \text{ non-zero } \mathbf{z} \in \mathcal{R}^{n-m}, \forall T \text{ with } |T| = \rho^*(\alpha, p)n, \\
& ||B_T\mathbf{z}||_p^p < \frac{1}{2}||B\mathbf{z}||_p^p) \\
= \; & P(\forall \mathbf{z} \in \mathcal{S}, \forall T \text{ with } |T| = \rho^*(\alpha, p)n, \\
& ||B_T\mathbf{z}||_p^p < \frac{1}{2}||B\mathbf{z}||_p^p) \\
\geq \; & P(\forall \mathbf{z} \in \mathcal{S}, \forall T \text{ with } |T| = \rho^*(\alpha, p)n, \\
& ||B_T\mathbf{z}||_p^p < \frac{1}{2}\lambda_{\min}(\alpha, p)n, \text{ and } ||B\mathbf{z}||_p^p > \lambda_{\min}(\alpha, p)n) \\
\geq \; & 1 - P(\exists \mathbf{z} \in \mathcal{S}, \text{ s.t. } ||B\mathbf{z}||_p^p \leq \lambda_{\min}(\alpha, p)n) \\
& -P(\exists \mathbf{z} \in \mathcal{S}, \exists T \text{ with } |T| = \rho^*(\alpha, p)n \text{ s.t.} \\
& ||B_T\mathbf{z}||_p^p \geq \lambda_{\min}(\alpha, p)n/2) \\
= \; & 1 - e^{-c_9 n} - e^{-c_{10} n}, \qquad\qquad (A.49)
\end{aligned}$$

where the first equality follows from Theorem 1, the second equality holds since for any non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$, $\mathbf{z}/||\mathbf{z}||_2 \in \mathcal{S}$. From Lemma 5 we know there exists $c_9 > 0$ such that $P(\exists \mathbf{z} \in \mathcal{S}, \text{ s.t. } ||B\mathbf{z}||_p^p \leq \lambda_{\min}(\alpha, p)n) \leq e^{-c_9 n}$, and from Lemma 6 we know there exists $c_{10} > 0$ such that $P(\exists \mathbf{z} \in \mathcal{S}, \exists T \text{ s.t. } ||B_T\mathbf{z}||_p^p \geq \frac{1}{2}\lambda_{\min}(\alpha, p)n) \leq e^{-c_{10} n}$, then there exists $c_{11} > 0$ which depends on $\alpha$, $p$ and $\lambda_{\min}$ such that the righthand side of (A.49) is greater than $1 - e^{-c_{11} n}$. Therefore, $\ell_p$-minimization can recover all the $\rho^*(\alpha, p)n$-sparse vectors with probability at least $1 - e^{-c_{11} n}$. □

## A.13 Proof of Lemma 7

*Proof.* Let $\alpha' = \frac{\alpha-\rho}{1-\rho}$. Define $c'_{\max} = \frac{1}{(1-\rho)n}\max_{\mathbf{z}\in\mathcal{S}} \|B_{T^c}\mathbf{z}\|_p^p$. Let $\Sigma_4$ be a $\gamma$-net of $\mathcal{S}$ with cardinality at most $(1+2/\gamma)^{n-m}$ and $\gamma$ be the value where $\lambda_{\max}(\alpha', p)$ is achieved in (A.34). We use $\lambda_{\max}$ to denote $\lambda_{\max}(\alpha', p)$ for simplicity here in the proof. Then from (A.34) we have

$$\lambda_{\max} = \hat{a}(\alpha', p, \gamma)/(1-\gamma^p), \tag{A.50}$$

where according to (A.33), $\hat{a}(\alpha', p, \gamma)$ has the property that

$$(1-\alpha')\log(1+\frac{2}{\gamma}) + \min_{t>0}(\log(E[e^{t|X|^p}]) - \hat{a}(\alpha', p, \gamma)t) < 0. \tag{A.51}$$

Combining (A.50) and (A.51), we have

$$(1-\alpha')\log(1+\frac{2}{\gamma}) + \min_{t>0}(\log(E[e^{t|X|^p}]) - \lambda_{\max}(1-\gamma^p)t) < 0. \tag{A.52}$$

Define

$$\eta' = \frac{1}{(1-\rho)n}\max_{\mathbf{z}\in\Sigma_4} \|B_{T^c}\mathbf{z}\|_p^p.$$

Then by arguments similar to those that lead to (A.30), we have

$$c'_{\max} \le \eta'/(1-\gamma^p).$$

We first show that with overwhelming probability, $\|B_{T^c}\mathbf{z}\|_p^p < (1-\rho)\lambda_{\max} n$ for all $\mathbf{z}$ in $\mathcal{S}$, or equivalently $c'_{\max} < \lambda_{\max}$. Note that

$$
\begin{aligned}
& P(c'_{\max} \geq \lambda_{\max}) \\
\leq\ & P(\eta'/(1-\gamma^p) \geq \lambda_{\max}) \\
=\ & P(\exists \mathbf{z} \in \Sigma_4 \text{ s.t. } \|B_{T^c}\mathbf{z}\|_p^p \geq (1-\rho)\lambda_{\max}(1-\gamma^p)n) \\
\leq\ & \sum_{\mathbf{z}\in\Sigma_4} P(\|B_{T^c}\mathbf{z}\|_p^p \geq (1-\rho)\lambda_{\max}(1-\gamma^p)n) \\
\leq\ & (1+\frac{2}{\gamma})^{n-m} \min_{t>0} \frac{E[e^{t\sum_{i\in T_c}|B_i\mathbf{z}|^p}]}{e^{t(1-\rho)\lambda_{\max}(1-\gamma^p)n}} \\
=\ & (1+\frac{2}{\gamma})^{(1-\alpha)n} \min_{t>0} \frac{(E[e^{t|X|^p}])^{(1-\rho)n}}{e^{t(1-\rho)\lambda_{\max}(1-\gamma^p)n}} \\
=\ & e^{(1-\rho)n(\frac{1-\alpha}{1-\rho}\log(1+\frac{2}{\gamma})+\min_{t>0}(\log(E[e^{t|X|^p}])-\lambda_{\max}(1-\gamma^p)t))} \\
=\ & e^{(1-\rho)n((1-\alpha')\log(1+\frac{2}{\gamma})+\min_{t>0}(\log(E[e^{t|X|^p}])-\lambda_{\max}(1-\gamma^p)t))}, \qquad \text{(A.53)}
\end{aligned}
$$

where $X \sim \mathcal{N}(0,1)$. Combining (A.52) and (A.53), we conclude that there exists $c_{12} > 0$ such that $P(c'_{\max} \geq \lambda_{\max}) \leq e^{-c_{12}n}$. Therefore with probability at least $1-e^{-c_{12}n}$, for all $\mathbf{z} \in \mathcal{S}$, $\|B_{T^c}\mathbf{z}\|_p^p < (1-\rho)\lambda_{\max}(\alpha', p)n$ holds.

Similarly, define $c'_{\min} = \frac{1}{(1-\rho)n}\min_{\mathbf{z}\in\mathcal{S}}\|B\mathbf{z}\|_p^p$. Let $\Sigma_5$ be a $\hat{\gamma}$-net of $\mathcal{S}$ with cardinality at most $(1+2/\hat{\gamma})^{n-m}$ and $\hat{\gamma}$ be the value where $\lambda_{\min}(\alpha', p)$ is achieved, note that from (A.44) we have

$$\lambda_{\min}(\alpha', p) = \hat{\gamma}^{p(1-\alpha'+\epsilon)} - \hat{\gamma}^p\lambda_{\max}(\alpha', p)$$

for some $\epsilon \in (0, \alpha')$. From (A.41) we also have that

$$(1-\alpha')\log(1+2/\hat{\gamma}) + \log(\Theta(\hat{\gamma}^{1-\alpha'+\epsilon})) + 1 < 0. \qquad \text{(A.54)}$$

We use $\lambda_{\min}$ and $\lambda_{\max}$ to denote $\lambda_{\min}(\alpha', p)$ and $\lambda_{\max}(\alpha', p)$ for simplicity. We define

$$\theta' = \frac{1}{(1-\rho)n}\min_{\mathbf{z}\in\Sigma_5}\|B_{T^c}\mathbf{z}\|_p^p.$$

Using the same arguments as those for (A.36), we have

$$c'_{\min} \geq \theta' - \gamma^p c'_{\max}.$$

We next show that with overwhelming probability, $\|B_{T^c}\mathbf{z}\|_p^p > (1-\rho)\lambda_{\min} n$ for all $\mathbf{z}$ in $\mathcal{S}$, or equivalently $c'_{\min} > \lambda_{\min}$. Note that the probability that $c'_{\min} \leq \lambda_{\min}$ is

$$\begin{aligned}
& P(c'_{\min} \leq \lambda_{\min}) \\
= \;& P(c'_{\min} \leq \hat{\gamma}^{p(1-\alpha'+\epsilon)} - \hat{\gamma}^p \lambda_{\max}) \\
\leq \;& P(\theta' - \hat{\gamma}^p c'_{\max} \leq \hat{\gamma}^{p(1-\alpha'+\epsilon)} - \hat{\gamma}^p \lambda_{\max}) \\
\leq \;& P(\theta' \leq \hat{\gamma}^{p(1-\alpha'+\epsilon)}) + P(c'_{\max} \geq \lambda_{\max}) \\
\leq \;& P(\theta' \leq \hat{\gamma}^{p(1-\alpha'+\epsilon)}) + e^{-c_{13} n},
\end{aligned} \tag{A.55}$$

where the last inequality follows from $P(c'_{\max} \geq \lambda_{\max}) \leq e^{-c_{12} n}$. To calculate $P(\theta' \leq \hat{\gamma}^{p(1-\alpha'+\epsilon)})$, note that

$$\begin{aligned}
& P(\theta' \leq \hat{\gamma}^{p(1-\alpha'+\epsilon)}) \\
= \;& P(\exists \mathbf{z} \in \Sigma_5 \text{ s.t. } \|B_{T^c}\mathbf{z}\|_p^p \leq (1-\rho)\hat{\gamma}^{p(1-\alpha'+\epsilon)} n) \\
\leq \;& \sum_{\mathbf{z} \in \Sigma_5} P(\sum_{i \in T^c} |B_i \mathbf{z}|^p \leq (1-\rho)\hat{\gamma}^{p(1-\alpha'+\epsilon)} n) \\
\leq \;& (1 + \frac{2}{\hat{\gamma}})^{(1-\alpha)n} e^{(1-\rho)n} (E[e^{-\hat{\gamma}^{-p(1-\alpha'+\epsilon)}|X|^p}])^{(1-\rho)n} \\
= \;& e^{(1-\rho)n((1-\alpha')\log(1+\frac{2}{\hat{\gamma}}) + \log(E[e^{-\hat{\gamma}^{-p(1-\alpha'+\epsilon)}|X|^p}]) + 1)} \\
= \;& e^{(1-\rho)n((1-\alpha')\log(1+\frac{2}{\hat{\gamma}}) + \log(\Theta(\hat{\gamma}^{1-\alpha'+\epsilon})) + 1)},
\end{aligned} \tag{A.56}$$

where $X \sim \mathcal{N}(0,1)$, the first inequality follows from the union bound, the second inequality follows from the Chernoff bound, and the last equality follows from (A.40). Combining (A.54) and (A.56), we have

$$P(\theta' \leq \hat{\gamma}^{p(1-\alpha'+\epsilon)}) \leq e^{-\kappa n}, \tag{A.57}$$

for some positive $\kappa > 0$. Thus, from (A.55) and (A.57) we have

$$P(c'_{\min} \leq \lambda_{\min}) \leq e^{-\kappa n} + e^{-c_{12} n} \leq e^{-c_{13} n},$$

for some $c_{13} > 0$. Then, with probability at least $1 - e^{-c_{13} n}$, for all $\mathbf{z} \in \mathcal{S}$, $\|B_{T^c}\mathbf{z}\|_p^p > (1-\rho)\lambda_{\min}(\alpha', p) n$. $\square$

## A.14 Calculation of $\tilde{\lambda}_{\max}(\alpha, p, \rho)$ in Lemma 8

*Proof.* Define $\tilde{c}_{\max} = \frac{1}{\rho n}\max_{\mathbf{z}\in\mathcal{S}} \|B_{T^-}\mathbf{z}\|_p^p$. Let $\Sigma_6$ be a $\gamma$-net of $\mathcal{S}$ with cardinality at most $(1+2/\gamma)^{n-m}$ and $\gamma > 0$ to be chosen later, and define $\tilde{\eta} = \frac{1}{\rho n}\max_{\mathbf{z}\in\Sigma_4} \|B_{T^-}\mathbf{z}\|_p^p$. Then from (A.21), for any $\mathbf{z}\in\mathcal{S}$, $\mathbf{z} = \sum_{j\geq 0}\gamma_j\mathbf{v}_j$ holds, where $\gamma_0 = 1$, $\gamma_j \leq \gamma^j$ and $\mathbf{v}_j \in \Sigma_6$. From (A.27) we have

$$\begin{aligned}
\|B_{T^-}\mathbf{z}\|_p^p &\leq \sum_{j\geq 0}\gamma^{jp}\sum_{i\in T: B_i\mathbf{v}_j<0}|B_i\mathbf{v}_j|^p \\
&\leq \sum_{j\geq 0}\gamma^{jp}\tilde{\eta}\rho n \\
&= \tilde{\eta}\rho n/(1-\gamma^p), \qquad \text{(A.58)}
\end{aligned}$$

where the second inequality follows from the definition of $\tilde{\eta}$. Since (A.58) holds for every $\mathbf{z}\in\mathcal{S}$, then $\tilde{c}_{\max}\rho n \leq \tilde{\eta}\rho n/(1-\gamma^p)$, which leads to $\tilde{c}_{\max} \leq \tilde{\eta}/(1-\gamma^p)$. For any given $\mathbf{z}\in\mathcal{S}$, define a random variable $S_i$ for each $i$ in $T$, and $S_i$ is equal to 1 if $B_i\mathbf{z} < 0$ and equal to 0 otherwise. Then $\|B_{T^-}\mathbf{z}\|_p^p = \sum_{i\in T}|B_i\mathbf{z}|^p S_i$.

Given $\gamma$, for any $\tilde{a}$, we will characterize the probability that $\tilde{c}_{\max}$ is greater than $\tilde{a}/(1-\gamma^p)$. We will find the smallest value of $\tilde{a}$ such that this probability still exponentially decays to zero, and take the corresponding $\tilde{a}/(1-\gamma^p)$ as an upper bound of $\tilde{c}_{\max}$. Note that

$$\begin{aligned}
& P(\tilde{c}_{\max} \geq \frac{\tilde{a}}{1-\gamma^p}) \leq P(\frac{\tilde{\eta}}{1-\gamma^p} \geq \frac{\tilde{a}}{1-\gamma^p}) \\
= \; & P(\tilde{\eta} \geq \tilde{a}) = P(\exists \mathbf{z}\in\Sigma_6 \text{ s.t. } \|B_{T^-}\mathbf{z}\|_p^p \geq \tilde{a}\rho n) \\
\leq \; & \sum_{\mathbf{z}\in\Sigma_6} P(\|B_{T^-}\mathbf{z}\|_p^p \geq \tilde{a}\rho n) \\
= \; & (1+\frac{2}{\gamma})^{n-m} P(\sum_{i\in T}|B_i\mathbf{z}|^p S_i \geq \tilde{a}\rho n) \\
\leq \; & (1+\frac{2}{\gamma})^{(1-\alpha)n}\min_{t>0}\frac{(E[e^{t|X|^pS}])^{\rho n}}{e^{t\tilde{a}\rho n}} \\
= \; & e^{((1-\alpha)\log(1+\frac{2}{\gamma})+\rho\min_{t>0}(\log(E[e^{t|X|^pS}])-\tilde{a}t))n}, \qquad \text{(A.59)}
\end{aligned}$$

where $X \sim \mathcal{N}(0,1)$, $S = 1$ if $X < 0$ and $S = 0$ otherwise.

To solve the minimization problem in the righthand side of (A.59), note that $\log(E[e^{t|X|^p S}])$ is the cumulant generating function and is convex [45] with respect to $t$, then $\log(E[e^{t|X|^p S}]) - \tilde{a}t$ is also convex, and its minimum is achieved where its first derivative with respect to $t$ is 0. Define $t^* := \arg\min_t [\log(E[e^{t|X|^p S}]) - \tilde{a}t]$, then we have

$$\begin{aligned} 0 &= \frac{d[\log(E[e^{t|X|^p S}]) - \tilde{a}t]}{dt}\Big|_{t=t^*} \\ &= \frac{E[|X|^p S\, e^{t^*|X|^p S}]}{E[e^{t^*|X|^p S}]} - \tilde{a}. \end{aligned} \tag{A.60}$$

(A.60) determines $t^*$ given $\tilde{a}$. The derivative of $t^*$ with respect to $\tilde{a}$ is

$$\frac{dt^*}{d\tilde{a}} = \left(\frac{d\tilde{a}}{dt^*}\right)^{-1} = \frac{(E[e^{t^*|X|^p S}])^2}{E[e^{t^*|X|^p S}]E[|X|^{2p} S^2 e^{t^*|X|^p S}] - (E[|X|^p S\, e^{t^*|X|^p S}])^2} > 0,$$

where the inequality follows from Cauchy-Schwarz inequality. Since when $\tilde{a} = E[|X|^p S]$, $t^* = 0$ from (A.60), then when $\tilde{a} > E[|X|^p S]$, we have $t^* > 0$. Thus $t^* = \min_{t>0}(\log(E[e^{t|X|^p}]) - at)$ when $\tilde{a} > E[|X|^p S]$. Given $\tilde{a}$, we can numerically compute $t^*$ by (A.60) and plug it into (A.59) to obtain an upper bound of $P(\tilde{c}_{\max} \geq \frac{\tilde{a}}{1-\gamma^p})$.

Then the question is how small can $\tilde{a}$ be while the exponent on the righthand side of (A.59) is still negative. Given $\gamma$, the exponent on the righthand side of (A.59) is negative when $\tilde{a}$ is large enough. To see this, note that if $t = 2(1-\alpha)\log(1+2/\gamma)/(\tilde{a}\rho)$, then $\log(E[e^{t|X|^p S}]) - \tilde{a}t$ goes to $-2(1-\alpha)\log(1+2/\gamma)/\rho$ as $\tilde{a}$ goes to infinity. Thus, when $\tilde{a}$ is sufficiently large, $\rho \min_{t>0}(\log(E[e^{t|X|^p S}]) - \tilde{a}t) < -(1-\alpha)\log(1+2/\gamma)$. Therefore, the exponent on the righthand side of (A.59) is negative when $\tilde{a}$ is large enough. Thus, we can pick $\overline{a}(\alpha, p, \rho, \gamma)$ such that the exponent on the righthand side of (A.59) is negative for all $a \geq \overline{a}(\alpha, p, \rho, \gamma)$, and positive for all $a \leq \overline{a}(\alpha, p, \rho, \gamma) - \epsilon$ for some small enough $\epsilon > 0$. Therefore

$$(1-\alpha)\log(1+\frac{2}{\gamma}) + \rho \min_{t>0}(\log(E[e^{t|X|^p S}]) - \overline{a}(\alpha, p, \rho, \gamma)t) < 0.$$

Then there exists some constant $c_{14} > 0$ such that

$$\begin{aligned} & P(\tilde{c}_{\max} \geq \frac{\overline{a}(\alpha, p, \rho, \gamma)}{1-\gamma^p}) \\ \leq\ & e^{((1-\alpha)\log(1+\frac{2}{\gamma})+\rho\min_{t>0}(\log(E[e^{t|X|^p S}])-\overline{a}(\alpha,p,\rho,\gamma)t))n} \\ =\ & e^{-c_{14}n}. \end{aligned}$$

Thus, for all $\gamma \in (0,1)$, $\hat{a}(\alpha, p, \rho, \gamma)\rho n/(1-\gamma^p)$ can be viewed as an upper bound of $\|B_{T^-}\mathbf{z}\|$ for all $\mathbf{z} \in \mathcal{S}$ in the sense that the probability that $\|B_{T^-}\mathbf{z}\|_p^p \geq \hat{a}(\alpha, p, \rho, \gamma)\rho n/(1-\gamma^p)$ for some $\mathbf{z} \in \mathcal{S}$ decays exponentially to zero. Since we would like such an upper bound to be as small as possible, we let

$$\tilde{\lambda}_{\max}(\alpha, p, \rho) = \min_{\gamma \in (0,1)} \frac{\overline{a}(\alpha, p, \rho, \gamma)}{1-\gamma^p}, \tag{A.61}$$

then with overwhelming probability, $\tilde{c}_{\max} < \tilde{\lambda}_{\max}(\alpha, p, \rho)$, or equivalently, for every $\mathbf{z} \in \mathcal{S}$, $\|B_{T^-}\mathbf{z}\|_p^p < (1-\rho)\tilde{\lambda}_{\max}(\alpha, p, \rho)n$. Thus, Lemma 8 follows. □

## A.15 Proof of Theorem 8

*Proof.* We first consider the case that there exists some $\rho_w^*(\alpha, p)$ (denoted by $\rho_w^*$ for simplicity here in this proof) such that $\rho_w^* > \rho^*(\alpha, p)$, where $\rho^*(\alpha, p)$ is the lower bound of strong threshold in Theorem 7, and the following inequality holds,

$$\rho_w^* \tilde{\lambda}_{\max}(\alpha, p, \rho_w^*) \leq (1-\rho_w^*)\lambda_{\min}(\frac{\alpha-\rho_w^*}{1-\rho_w^*}, p). \tag{A.62}$$

We will show that such $\rho_w^*$ indeed has the property that Theorem 8 states, i.e. it is a lower bound of weak recovery threshold.

Now consider the probability that $\ell_p$-minimization can recover all the $\rho_w^* n$-sparse $\mathbf{x}$ on one fixed support $T$ with one fixed sign pattern. From Theorem 3 we know that $\|B_{T^-}\mathbf{z}\|_p^p < \|B_{T^c}\mathbf{z}\|_p^p$ for all non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$ is a sufficient condition for the success of weak recovery, thus

$$
\begin{aligned}
& P(\text{Weak recovery succeeds up to } \rho_w^* n\text{-sparse}) \\
\geq\ & P(\forall \text{ non-zero } \mathbf{z} \in \mathcal{R}^{n-m}, \|B_{T^-}\mathbf{z}\|_p^p < \|B_{T^c}\mathbf{z}\|_p^p) \\
=\ & P(\forall \mathbf{z} \in \mathcal{S}, \|B_{T^-}\mathbf{z}\|_p^p < \|B_{T^c}\mathbf{z}\|_p^p) \\
\geq\ & P(\forall \mathbf{z} \in \mathcal{S}, \|B_{T^-}\mathbf{z}\|_p^p < \rho_w^* \tilde{\lambda}_{\max}(\alpha, p, \rho_w^*), \text{ and} \\
& \|B_{T^c}\mathbf{z}\|_p^p > (1-\rho_w^*)\lambda_{\min}(\frac{\alpha - \rho_w^*}{1-\rho_w^*}, p)) \\
\geq\ & 1 - e^{-c_{14}n} - e^{-c_{13}n},
\end{aligned} \tag{A.63}
$$

where the first equality holds since for any non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$, $\mathbf{z}/\|\mathbf{z}\|_2 \in \mathcal{S}$, and the second inequality follows from (A.62). From Lemma 7 we know there exists $c_{13} > 0$ such that $P(\forall \mathbf{z} \in \mathcal{S}, \|B_{T^c}\mathbf{z}\|_p^p > (1-\rho_w^*)\lambda_{\min}(1 - \frac{1-\alpha}{1-\rho_w^*}, p)) \geq 1 - e^{-c_{13}n}$, and from Lemma 8 we know there exists $c_{14} > 0$ such that $P(\forall \mathbf{z} \in \mathcal{S}, \|B_{T^-}\mathbf{z}\|_p^p < \rho_w^* \tilde{\lambda}_{\max}(\alpha, p, \rho_w^*)) \geq 1 - e^{-c_{14}n}$, then the third inequality of (A.63) holds from the union bound. Thus, there exists $c_{15} > 0$ such that with probability at least $1 - e^{-c_{15}n}$, $\ell_p$-minimization problem can recover all $\rho_w^* n$-sparse vectors on fixed support $T$ with fixed sign pattern, then Theorem 8 holds.

Now consider the case that there is no $\rho_w^* > \rho^*(\alpha, p)$ satisfying (A.62), where $\rho^*(\alpha, p) > 0$ is the lower bound of strong threshold in Theorem 7, then we can simply define $\rho_w^*(\alpha, p) := \rho^*(\alpha, p)$. Since $\rho_w^*(\alpha, p)$ is a lower bound of strong threshold and then a lower bound of weak threshold, thus Theorem 8 follows. □

APPENDIX B

**PROOFS OF CHAPTER 2**

## B.1 Proof of Theorem 12

Let $A^{m\times n}$ denote the matrix with $m$ realizations of the $n$-step Markov chain. To prove the statement, from [20], we only need to show that the probability that every $2k$ columns of $A$ are linearly independent goes to 1 as $n$ goes to infinity.

Let $A_I$ be a submatrix of $A$ with columns in $I$, where $I$ is an index set with $|I| = 2k$. Let $A_{S_jI}$ $(1 \leq j \leq \lfloor \frac{m}{2k} \rfloor)$ be a submatrix of $A_I$ formed by row $2k(j-1)+1$ to row $2kj$ of $A_I$. Given $I$, the probability that rank($A_{S_jI}$)$< 2k$ is the same for every given $j$, and let it denoted by $\pi_d^I$. Let $P_d^I$ denote the probability that rank($A_I$)$< 2k$, then

$$P_d^I \leq (\pi_d^I)^{\lfloor \frac{m}{2k} \rfloor}. \tag{B.1}$$

To characterize $\pi_d^I$, consider matrix $B^{2k\times 2k}$ with $B_{ii} = 0$ for $i = 2, 3, ..., 2k$ and $B_{ij} = 1$ for all the other elements. Since rank($B$)$= 2k$, then

$$\pi_d^I \leq 1 - P(A_{S_jI} \text{ is a row permutation of } B). \tag{B.2}$$

One can check that in this Markov chain, for every $1 \leq i < k \leq n$, $P(X_k = 1 \mid X_i = 1) \geq 1/2$, $P(X_k = 0 \mid X_i = 1) \geq 1/4$, $P(X_k = 1 \mid X_i = 0) \geq 1/2$, and $P(X_k = 1) \geq 1/2$. Since $B$ has $(2k)!$ different row permutations,

$$P(A_{S_jI} \text{ is a row permutation of } B) \geq (2k)!/2^{4k^2+2k-1}. \tag{B.3}$$

Combining (B.1), (B.2) and (B.3), we have

$$\begin{aligned}
&P(\text{every } 2k \text{ columns of } A \text{ are linearly independent})\\
=&1 - P(\text{rank}(A_I) < 2k \text{ for some } I \text{ with } |I| = 2k)\\
\geq&1 - \binom{n}{2k} P_d^I \geq 1 - \binom{n}{2k} e^{-(2k)!(\frac{1}{2})^{4k^2+2k-1}\lfloor \frac{m}{2k} \rfloor},
\end{aligned}$$

where the first inequality follows from the union bound. Then if $m = g(k)\log n = (2k+1)2^{4k^2+2k-1}\log n/(2k-1)!$, the probability that every $2k$ columns of $A$ are linearly independent is at least $1 - 1/((2k)!n)$.

## B.2 Proof of Proposition 2

We view nodes $2i-1$ and $2i$ as a group for every $i$ ($1 \leq i \leq \lfloor \frac{n}{2} \rfloor$), denoted by $B_i$. Consider the special case that for some $t$, both nodes in $B_t$ are '1's, and all other nodes are '0's. Then every measurement that passes either node or both nodes in $B_t$ is always '1'. Consider a reduced graph with $B_i$, $\forall i$ as nodes, and link $(B_i, B_j)$ $(i \neq j)$ exists only if in $\mathcal{G}^4$ there is a path from a node in $B_i$ to a node in $B_j$ without going though any other node not in $B_i$ or $B_j$. $B_i$ is '1' if both node $2i-1$ and node $2i$ in $\mathcal{G}^4$ are '1's and is '0' otherwise. The reduced network is a ring with $\lfloor \frac{n}{2} \rfloor$ nodes, and thus $\lfloor n/4 \rfloor$ measurements are required to locate one non-zero element in the reduced network. Then only to locate two consecutive non-zero elements associated with $\mathcal{G}^4$, we need at least $\lfloor n/4 \rfloor$ measurements, and the claim follows.

## B.3 Proof of Theorem 15

Since checking whether or not $r$ given sets form an $r$-partition takes polynomial time, $r$-partition problem is NP.

We next show that the NP-complete $r$-coloring ($r \geq 3$) problem is polynomial time reducible to $r$-partition problem.

Let $G = (V, E)$ and an integer $r$ be an instance of $r$-coloring. For every $(u, v) \in E$, add a node $w$ and two links $(w, u)$ and $(w, v)$. Let $W$ denote the set of nodes added. Add a link between every pair of nodes in $V$ not already joined by a link. Let $H$ denote the augmented graph and let $V' = V \cup W$. We claim that if there exists an $r$-partition of $H$, then we can obtain an $r$-coloring of $G$, and vice versa.

Let $S_i$ ($i = 1, ..., r$) be an $r$-partition of $H$. Suppose there exists link $(u, v) \in E$ s.t. $u$ and $v$ both belong to $S_i$ for some $i$. Let $w$ denote the node in $W$ that only directly connects to $u$ and $v$. If $w \in S_i$, then $w$ has both neighbors in the same set with $w$, contradicting the definition of $r$-partition. If $w \notin S_i$, then $H_{V' \backslash S_i}$ is disconnected since $w$ does not connect to any node in $V' \backslash S_i$. It also contradicts the definition of $r$-partition. Thus, for every $(u, v) \in E$, node $u$ and $v$ belong to two sets $S_i$ and $S_j$ with $i \neq j$. Then we obtain an $r$-coloring of $G$.

Let $C_i \subset V$ ($i = 1, ..., r$) denote an $r$-coloring of $G$. We claim that $N_i = C_i$ ($i = 1, ..., r - 1$), and $N_r = C_r \cup W$ form an $r$-partition of $H$. First note for every $u \in V$, at least one of its neighbors is not in the same set as $u$. For every $w \in W$, $w$ is directly connected to $u$ and $v$ for some $(u, v) \in E$, and $u$ and $v$ are in different sets $C_i$ and $C_j$ for some $i \neq j$. Therefore, $w$ has at least one neighbor that is not in $N_r$. Second, we will show $H_{V' \backslash N_i}$ is connected for all $i$. $H_{V' \backslash N_r}$ is a complete subgraph, and thus connected. For every $i < r$, let $S_i := V \backslash C_i$, then $V' \backslash N_i = S_i \cup W$. $H_{S_i}$ is a complete subgraph, and thus connected. For every $w \in W$, since its two neighbors cannot be both in $C_i$, then at least one neighbor belongs to $S_i$, thus $H_{V' \backslash N_r} = H_{S_i \cup W}$ is connected. $N_i$ $(i = 1, ..., r)$ thus forms an $r$-partition.

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